First-order logic

Chapter 8

Outline

- ♢ Why FOL?
- ♢ Syntax and semantics of FOL
- ♢ Fun with sentences
- ♢ Wumpus world in FOL

Pros and cons of propositional logic

☺ Propositional logic is **declarative**: pieces of syntax correspond to facts

|--> knowledge and inference are separate. Inference is domain-independent.

☺ Propositional logic allows partial/disjunctive/negated information
(unlike most data structures and databases)

☺ Propositional logic is **compositional**:
meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$

☺ Meaning in propositional logic is **context-independent**
(unlike natural language, where meaning depends on context)

☹ Propositional logic has very limited expressive power
(unlike natural language)
E.g., cannot say "pits cause breezes in adjacent squares"
except by writing one sentence for each square

Procedural approach: programs represent the computational processes. Data structures within programs can represent the facts. Lack any general mechanisms for deriving facts from other facts: each update to a data structure is done by a domain-specific procedure.

Natural languages are very expressive! but ambigious & context dependent.

First-order logic

Whereas propositional logic assumes world contains **facts**,
first-order logic (like natural language) assumes the world contains

- Objects: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries . . .
- Relations: red, round, bogus, prime, multistoried . . .,
  brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, . . .
- Functions: father of, best friend, third inning of, one more than, end of . . .

e.g. "one plus two equals three"
- objects?
- relations?
- functions?

Logics in general

Language	Ontological Commitment	Epistemological Commitment
Propositional logic First-order logic Temporal logic Probability theory	facts	true/false/unknown
Fuzzy logic	facts + degree of truth	known interval value

- Ontology often deals with questions concerning what entities exist or may be said to exist and how such entities may be grouped, related within a hierarchy, and subdivided according to similarities and differences. (wiki)

- Epistomology: the study or a theory of the nature and grounds of knowledge especially with reference to its limits and validity. (merriam-webster)

Logics in general

Language	Ontological Commitment	Epistemological Commitment
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief
Fuzzy logic	facts + degree of truth	known interval value

Syntax of FOL: Basic elements

Constants	$KingJohn,\ 2,\ UCB, \ldots$
Predicates	$Brother,\ >, \ldots$
Functions	$Sqrt,\ LeftLegOf, \ldots$
Variables	$x,\ y,\ a,\ b, \ldots$
Connectives	$\wedge\ \vee\ \neg\ \Rightarrow\ \Leftrightarrow$
Equality	$=$
Quantifiers	$\forall\ \exists$

Syntax of FOL: Basic elements

$$
\begin{array}{rcl}
\mathit{Sentence} & \rightarrow & \mathit{AtomicSentence} \mid \mathit{ComplexSentence} \\
\mathit{AtomicSentence} & \rightarrow & \mathit{Predicate} \mid \mathit{Predicate}(\mathit{Term}, \ldots) \mid \mathit{Term} = \mathit{Term} \\
\mathit{ComplexSentence} & \rightarrow & (\ \mathit{Sentence}\ ) \mid [\ \mathit{Sentence}\ ] \\
 & \mid & \neg\ \mathit{Sentence} \\
 & \mid & \mathit{Sentence} \wedge \mathit{Sentence} \\
 & \mid & \mathit{Sentence} \vee \mathit{Sentence} \\
 & \mid & \mathit{Sentence} \Rightarrow \mathit{Sentence} \\
 & \mid & \mathit{Sentence} \Leftrightarrow \mathit{Sentence} \\
 & \mid & \mathit{Quantifier}\ \mathit{Variable}, \ldots\ \mathit{Sentence} \\
 & & \\
\mathit{Term} & \rightarrow & \mathit{Function}(\mathit{Term}, \ldots) \\
 & \mid & \mathit{Constant} \\
 & \mid & \mathit{Variable} \\
 & & \\
\mathit{Quantifier} & \rightarrow & \forall \mid \exists \\
\mathit{Constant} & \rightarrow & A \mid X_1 \mid \mathit{John} \mid \cdots \\
\mathit{Variable} & \rightarrow & a \mid x \mid s \mid \cdots \\
\mathit{Predicate} & \rightarrow & \mathit{True} \mid \mathit{False} \mid \mathit{After} \mid \mathit{Loves} \mid \mathit{Raining} \mid \cdots \\
\mathit{Function} & \rightarrow & \mathit{Mother} \mid \mathit{LeftLeg} \mid \cdots
\end{array}
$$

OPERATOR PRECEDENCE : $\neg, =, \wedge, \vee, \Rightarrow, \Leftrightarrow$

Atomic sentences

Atomic sentence $=$ $predicate(term_1, \ldots, term_n)$
or $term_1 = term_2$

Term $=$ $function(term_1, \ldots, term_n)$
or $constant$ or $variable$

E.g., $Brother(KingJohn, RichardTheLionheart)$
$> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$

A term with no variables is called a ground term.

Complex sentences

Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$$

E.g. $Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$

$>(1,2) \vee \leq(1,2)$

$>(1,2) \wedge \neg>(1,2)$

The semantics must relate sentences to models in order to determine truth.

Truth in first-order logic

Sentences are true with respect to a model and an interpretation

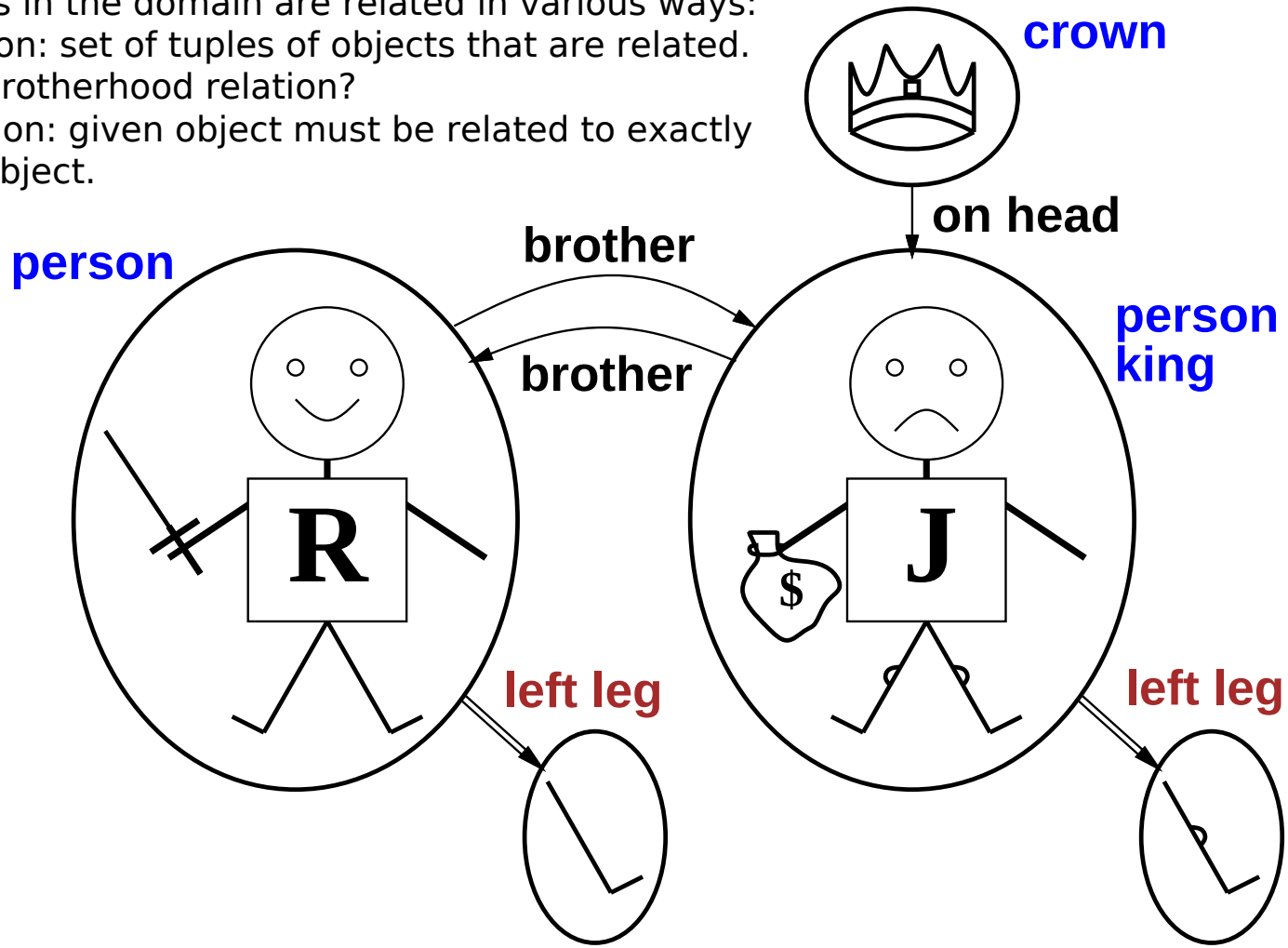
Model contains ≥ 1 objects (domain elements) and relations among them

Interpretation specifies referents for
- constant symbols $\rightarrow$ objects
- predicate symbols $\rightarrow$ relations
- function symbols $\rightarrow$ functional relations

An atomic sentence $predicate(term_1, \ldots, term_n)$ is true
iff the objects referred to by $term_1, \ldots, term_n$
are in the relation referred to by $predicate$

Models for FOL: Example

Domain for FOL is the set of objects it contains.
Objects in the domain are related in various ways:
- relation: set of tuples of objects that are related.
  e.g. brotherhood relation?
- function: given object must be related to exactly one object.



Relation: the set of tuples of objects that are related.
Objects?
Relations? Unary relations, binary relations?
Functions?

Truth example

Consider the interpretation in which
$Richard$ $\longrightarrow$ Richard the Lionheart
$John$ $\longrightarrow$ the evil King John
$Brother$ $\longrightarrow$ the brotherhood relation

Under this interpretation, $Brother(Richard, John)$ is true just in case Richard the Lionheart and the evil King John are in the brotherhood relation in the model

Models for FOL: Lots!

Entailment in propositional logic can be computed by enumerating models

We **can** enumerate the FOL models for a given KB vocabulary:

For each number of domain elements n from 1 to ∞
    For each k-ary predicate P_k in the vocabulary
        For each possible k-ary relation on n objects
            For each constant symbol C in the vocabulary
                For each choice of referent for C from n objects . . .

Computing entailment by enumerating FOL models is not easy!

Universal quantification

$\forall \langle variables \rangle \ \langle sentence \rangle$

Everyone at Berkeley is smart:
$\forall x \ \ At(x, Berkeley) \Rightarrow Smart(x)$

$\forall x \ \ P$ is true in a model m iff P is true with x being **each** possible object in the model

Roughly speaking, equivalent to the conjunction of instantiations of P

$$\begin{aligned} & (At(KingJohn, Berkeley) \Rightarrow Smart(KingJohn)) \\ \wedge \ & (At(Richard, Berkeley) \Rightarrow Smart(Richard)) \\ \wedge \ & (At(Berkeley, Berkeley) \Rightarrow Smart(Berkeley)) \\ \wedge \ & \ldots \end{aligned}$$

Quiz

- What is the interpretation of

$$\forall x \; At(x, Bogazici) \wedge Smart(x)$$

- Feedback:
  - https://tinyurl.com/yclbjjv6

A common mistake to avoid

Typically, $\Rightarrow$ is the main connective with $\forall$

Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall x \;\; At(x, Berkeley) \wedge Smart(x)$$

means "Everyone is at Berkeley and everyone is smart"

Existential quantification

$\exists \langle variables \rangle \ \langle sentence \rangle$

Someone at Stanford is smart:
$\exists x \ \ At(x, Stanford) \wedge Smart(x)$

$\exists x \ \ P$ is true in a model m iff P is true with x being
some possible object in the model

Roughly speaking, equivalent to the disjunction of instantiations of P

$$\begin{aligned} & (At(KingJohn, Stanford) \wedge Smart(KingJohn)) \\ \vee \ & (At(Richard, Stanford) \wedge Smart(Richard)) \\ \vee \ & (At(Stanford, Stanford) \wedge Smart(Stanford)) \\ \vee \ & \ldots \end{aligned}$$

Another common mistake to avoid

Typically, $\wedge$ is the main connective with $\exists$

Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

$$\exists x \;\; At(x, Stanford) \Rightarrow Smart(x)$$

is true if there is anyone who is not at Stanford!

Properties of quantifiers

$\forall x \ \forall y$ is the same as $\forall y \ \forall x$

$\exists x \ \exists y$ is the same as $\exists y \ \exists x$

$\exists x \ \forall y$ is **not** the same as $\forall y \ \exists x$

$\exists x \ \forall y \ \ Loves(x, y)$

$\forall y \ \exists x \ \ Loves(x, y)$

Quantifier duality: each can be expressed using the other

$\forall x \ \ Likes(x, IceCream)$ $\quad\quad$ $\neg\exists x \ \neg Likes(x, IceCream)$

$\exists x \ \ Likes(x, Broccoli)$ $\quad\quad$ $\neg\forall x \ \neg Likes(x, Broccoli)$

Note: P(x,y) is interpreted as "x is a P of y".

Properties of quantifiers

$\forall x \ \forall y$ is the same as $\forall y \ \forall x$

$\exists x \ \exists y$ is the same as $\exists y \ \exists x$

$\exists x \ \forall y$ is **not** the same as $\forall y \ \exists x$

$\exists x \ \forall y \ Loves(x, y)$

"There is a person who loves everyone in the world"

$\forall y \ \exists x \ Loves(x, y)$

"Everyone in the world is loved by at least one person"

Quantifier duality: each can be expressed using the other

$\forall x \ Likes(x, IceCream)$ $\quad\quad$ $\neg\exists x \ \neg Likes(x, IceCream)$

$\exists x \ Likes(x, Broccoli)$ $\quad\quad$ $\neg\forall x \ \neg Likes(x, Broccoli)$

relate to De Morgan's rule?

Fun with sentences

Brothers are siblings

Fun with sentences

Brothers are siblings

$\forall\, x, y \;\; Brother(x, y) \;\Rightarrow\; Sibling(x, y)$.

"Sibling" is symmetric

Fun with sentences

Brothers are siblings

$\forall\, x, y \;\; Brother(x, y) \;\Rightarrow\; Sibling(x, y)$.

"Sibling" is symmetric

$\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; Sibling(y, x)$.

One's mother is one's female parent

Fun with sentences

Brothers are siblings

$\forall\, x, y \;\; Brother(x, y) \;\Rightarrow\; Sibling(x, y)$.

"Sibling" is symmetric

$\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; Sibling(y, x)$.

One's mother is one's female parent

$\forall\, x, y \;\; Mother(x, y) \;\Leftrightarrow\; (Female(x) \wedge Parent(x, y))$.

A first cousin is a child of a parent's sibling

Fun with sentences

Brothers are siblings

$\forall\, x, y \;\; Brother(x, y) \;\Rightarrow\; Sibling(x, y)$.

"Sibling" is symmetric

$\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; Sibling(y, x)$.

One's mother is one's female parent

$\forall\, x, y \;\; Mother(x, y) \;\Leftrightarrow\; (Female(x) \wedge Parent(x, y))$.

A first cousin is a child of a parent's sibling

$\forall\, x, y \;\; FirstCousin(x, y) \;\Leftrightarrow\; \exists\, p, ps \;\; Parent(p, x) \wedge Sibling(ps, p) \wedge Parent(ps, y)$

Fun with sentences i.e. Kinship domain

$$\forall\, m, c \;\; Mother(c) = m \;\Leftrightarrow\; Female(m) \wedge Parent(m, c) \;.$$

One's husband is one's male spouse:

$$\forall\, w, h \;\; Husband(h, w) \;\Leftrightarrow\; Male(h) \wedge Spouse(h, w) \;.$$

Male and female are disjoint categories:

$$\forall\, x \;\; Male(x) \;\Leftrightarrow\; \neg Female(x) \;.$$

Parent and child are inverse relations:

$$\forall\, p, c \;\; Parent(p, c) \;\Leftrightarrow\; Child(c, p) \;.$$

A grandparent is a parent of one's parent:

$$\forall\, g, c \;\; Grandparent(g, c) \;\Leftrightarrow\; \exists\, p \;\; Parent(g, p) \wedge Parent(p, c) \;.$$

A sibling is another child of one's parents:

$$\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; x \neq y \wedge \exists\, p \;\; Parent(p, x) \wedge Parent(p, y) \;.$$

Definitions "bottom-out" at a basic set of predicates.
What is the basic set here?

Domain of natural numbers and addition

$$NatNum(0)\ .$$
$$\forall n\ \ NatNum(n)\ \Rightarrow\ NatNum(S(n))\ .$$

$$\forall n\ \ 0 \neq S(n)\ .$$
$$\forall m, n\ \ m \neq n\ \Rightarrow\ S(m) \neq S(n)\ .$$

$$\forall m\ \ NatNum(m)\ \Rightarrow\ \ +(0, m) = m\ .$$
$$\forall m, n\ \ NatNum(m) \wedge NatNum(n)\ \Rightarrow\ \ +(S(m), n) = S(+(m, n))$$

.. interpretation with infix notation ?

Domain of sets and operations

1. The only sets are the empty set and those made by adjoining something to a set:

$$\forall s \;\; Set(s) \;\Leftrightarrow\; (s = \{\}) \vee (\exists x, s_2 \;\; Set(s_2) \wedge s = \{x|s_2\}) \; .$$

2. The empty set has no elements adjoined into it. In other words, there is no way to decompose $\{\}$ into a smaller set and an element:

$$\neg\exists x, s \;\; \{x|s\} = \{\} \; .$$

3. Adjoining an element already in the set has no effect:

$$\forall x, s \;\; x \in s \;\Leftrightarrow\; s = \{x|s\} \; .$$

4. The only members of a set are the elements that were adjoined into it. We express this recursively, saying that x is a member of s if and only if s is equal to some set s_2 adjoined with some element y, where either y is the same as x or x is a member of s_2:

$$\forall x, s \;\; x \in s \;\Leftrightarrow\; \exists y, s_2 \;\; (s = \{y|s_2\} \wedge (x = y \vee x \in s_2)) \; .$$

5. A set is a subset of another set if and only if all of the first set's members are members of the second set:

$$\forall s_1, s_2 \;\; s_1 \subseteq s_2 \;\Leftrightarrow\; (\forall x \;\; x \in s_1 \;\Rightarrow\; x \in s_2) \; .$$

6. Two sets are equal if and only if each is a subset of the other:

$$\forall s_1, s_2 \;\; (s_1 = s_2) \;\Leftrightarrow\; (s_1 \subseteq s_2 \wedge s_2 \subseteq s_1) \; .$$

7. An object is in the intersection of two sets if and only if it is a member of both sets:

$$\forall x, s_1, s_2 \;\; x \in (s_1 \cap s_2) \;\Leftrightarrow\; (x \in s_1 \wedge x \in s_2) \; .$$

8. An object is in the union of two sets if and only if it is a member of either set:

$$\forall x, s_1, s_2 \;\; x \in (s_1 \cup s_2) \;\Leftrightarrow\; (x \in s_1 \vee x \in s_2) \; .$$

Equality

$term_1 = term_2$ is true under a given interpretation
if and only if $term_1$ and $term_2$ refer to the same object

$\exists\, x, y \;\; Brother(x, Richard) \wedge Brother(y, Richard) \wedge \neg(x = y)$

entence

$\exists\, x, y \;\; Brother(x, Richard) \wedge Brother(y, Richard)$

E.g., definition of (full) $Sibling$ in terms of $Parent$:

$$\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; [\neg(x = y) \wedge \exists\, m, f \;\; \neg(m = f) \wedge Parent(m, x) \wedge Parent(f, x) \wedge Parent(m, y) \wedge Parent(f, y)]$$

Actions: Turn(Right), Turn(Left), Forward, Shoot

Interacting with FOL KBs

Suppose a wumpus-world agent is using an FOL KB
and perceives a smell and a breeze (but no glitter) at $t = 5$:

$Tell(KB, Percept([Smell, Breeze, None], 5))$ ... add sentences to KB (assertion)
$Ask(KB, \exists a \;\; Action(a, 5))$ ... ask questions

constant, variable, relation, function?

I.e., does KB entail any particular actions at $t = 5$?

Answer: $Yes, \; \{a/Shoot\}$ $\leftarrow$ substitution (binding list)

Given a sentence S and a substitution σ,
$S\sigma$ denotes the result of plugging σ into S; e.g.,
$S = Smarter(x, y)$
$\sigma = \{x/Hillary, y/Bill\}$
$S\sigma = Smarter(Hillary, Bill)$

$Ask(KB, S)$ returns some/all σ such that $KB \models S\sigma$

Knowledge base for the wumpus world

"Perception"

$\forall\, b, g, t \;\; Percept([Smell, b, g], t) \;\Rightarrow\; Smelt(t)$

$\forall\, s, b, t \;\; Percept([s, b, Glitter], t) \;\Rightarrow\; AtGold(t)$

Reflex: $\forall\, t \;\; AtGold(t) \;\Rightarrow\; Action(Grab, t)$

Reflex with internal state: do we have the gold already?

$\forall\, t \;\; AtGold(t) \wedge \neg Holding(Gold, t) \;\Rightarrow\; Action(Grab, t)$

$Holding(Gold, t)$ cannot be observed

$\Rightarrow$ keeping track of change is essential

Deducing hidden properties

Properties of locations:

$\forall x, t \;\; At(Agent, x, t) \wedge Smelt(t) \;\Rightarrow\; Smelly(x)$

$\forall x, t \;\; At(Agent, x, t) \wedge Breeze(t) \;\Rightarrow\; Breezy(x)$

Squares are breezy near a pit:

Wumpus(x), Wall(x), Pit(x), Adjacent(x,y)

Diagnostic rule—infer cause from effect

$\forall y \;\; Breezy(y) \;\Rightarrow$???

Causal rule—infer effect from cause

??? $\Rightarrow\; Breezy(y)$

Definition for the $Breezy$ predicate:

$\forall y \;\; Breezy(y) \;\Leftrightarrow$???

Deducing hidden properties

Properties of locations:

$\forall x, t \;\; At(Agent, x, t) \wedge Smelt(t) \;\Rightarrow\; Smelly(x)$

$\forall x, t \;\; At(Agent, x, t) \wedge Breeze(t) \;\Rightarrow\; Breezy(x)$

Squares are breezy near a pit:

Pit(x), Adjacent(x,y), Wumpus(x), Wall(x)

Diagnostic rule—infer cause from effect

$$\forall y \;\; Breezy(y) \;\Rightarrow\; \exists x \;\; Pit(x) \wedge Adjacent(x, y)$$

Causal rule—infer effect from cause

$$\forall x, y \;\; Pit(x) \wedge Adjacent(x, y) \;\Rightarrow\; Breezy(y)$$

Neither of these is complete—e.g., the causal rule doesn't say whether squares far away from pits can be breezy

Definition for the $Breezy$ predicate:

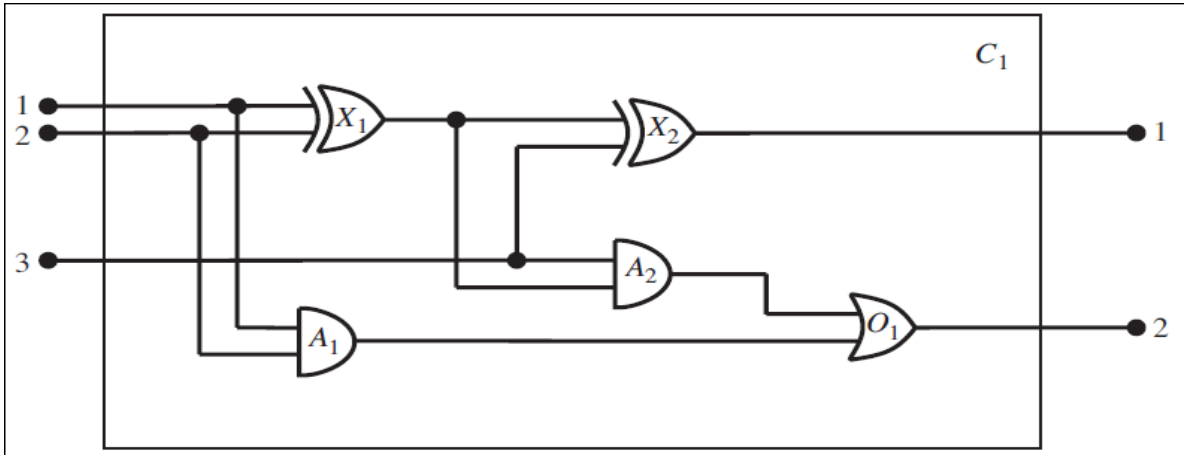
$$\forall y \;\; Breezy(y) \;\Leftrightarrow\; [\exists x \;\; Pit(x) \wedge Adjacent(x, y)]$$

Knowledge Engineering in FOL

Create a formal representation of the objects and relations in the domain.

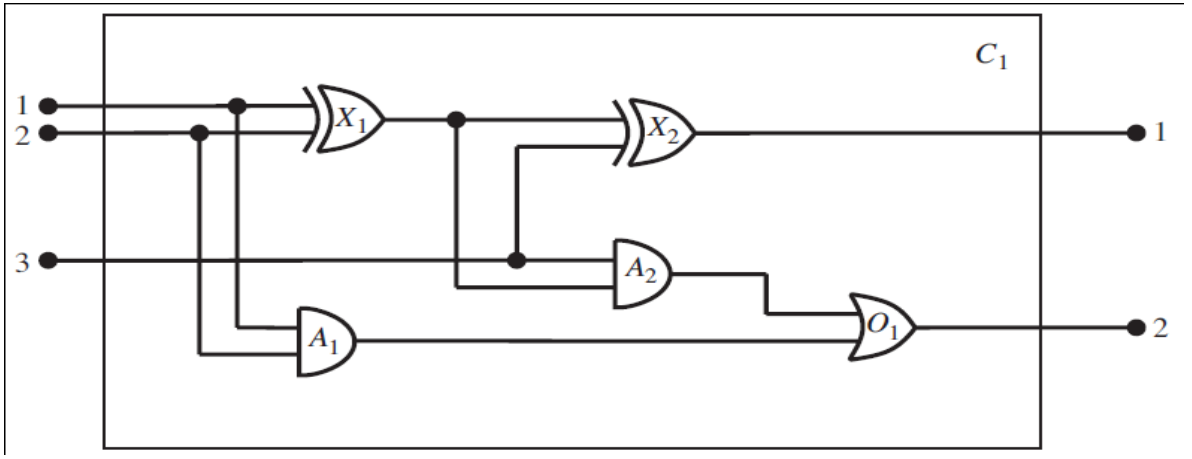
- Identify the task: The range of questions, kinds of facts.
- Assemble relevant knowledge: How the domain works
- Decide on vocabulary of predicates, functions, constants.
- Encode general knowledge: Write down the axioms for all vocabulary terms.
- Encode a description of the specific problem instance
- Pose queries to the inference procedure
- Debug the KB

Knowledge Engineering in FOL



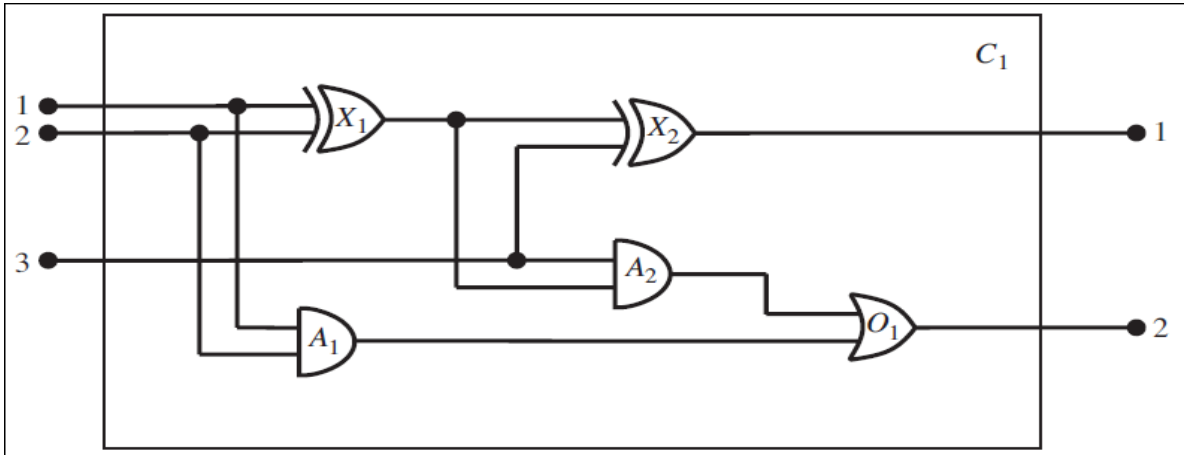
- Identify task: Assemble relevant knowledge:
- Decide on vocabulary:
  - Distinguish gates,
  - Gate behavior:
  - Terminals:
  - *Value of terminal:*
  - Connectivity:
- *Encode general knowledge of the domain:*
- *Encode specific problem instance:*

Knowledge Engineering in FOL



- Identify task: Input → Output
- Assemble relevant knowledge: 4 types of gates: XOR, AND, OR
- Decide on vocabulary:
  - Distinguish gates, X_1, X_2
  - Gate behavior: Type (X_1)=XOR or Type(X_1, XOR) or XOR(X_1)
  - Terminals: *In(1,X_2), Out(1, X_1)*
    - *Signal*
  - Connectivity: *Connected(Out(1,X_1), In(1,X_2))*

Knowledge Engineering in FOL



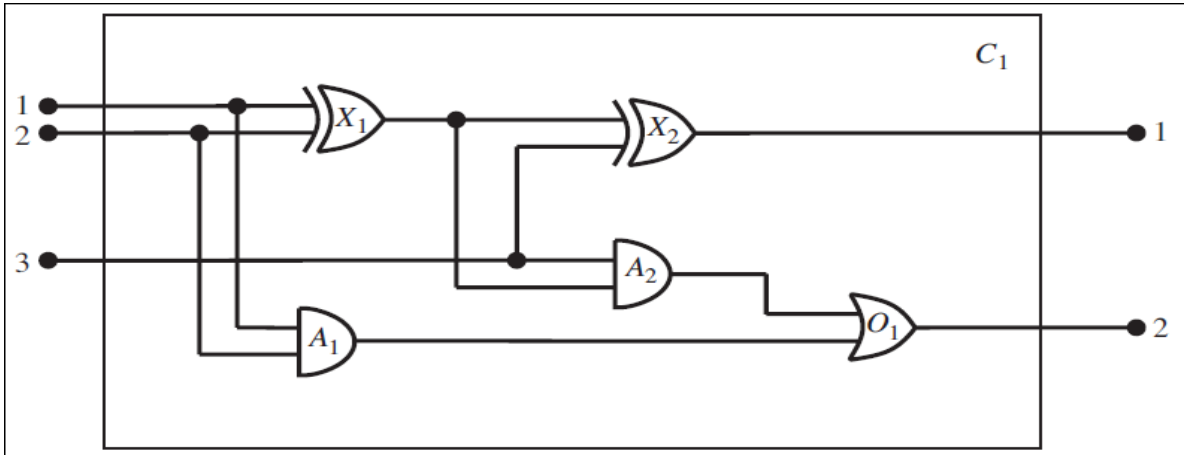
- Encode general knowledge of the domain:

$$\forall t_1, t_2 \; Connected(t_1, t_2) \Rightarrow Signal(t_1) = Signal(t_2)$$
$$\forall t \; Signal(t) = 1 \vee Signal(t) = 0, 1 \neq 0$$
$$\forall g \; Type(g) = OR \Rightarrow Signal(Out(1, g)) = 1 \Leftrightarrow \exists n \; Signal(In(n, g)) = 1$$

- Encode specific problem instance:

$$\text{Type}(\text{X}_1) = XOR \; Type(X_1) = AND ...$$
$$\text{Connected } (\text{Out}(1,\text{X}_1), In(1, X_2))$$
$$\text{Connected } (\text{In } (1,\text{C}_1), In(1, X_1))$$
..

Knowledge Engineering in FOL



- Pose queries:

$\exists i_1, i_2, i_3$

$\text{Signal}(\text{In}(1,\text{C}_1)) = i_1 \wedge Signal(In(2, C_1)) = i_2 \wedge Signal(In(3, C_1)) = i_3 \wedge Signal(Out(1, C_1)) = 0 \wedge Signal(Out(2, C_1)) = 1$

Summary

First-order logic:

- objects and relations are semantic primitives
- syntax: constants, functions, predicates, equality, quantifiers

Increased expressive power: sufficient to define wumpus world

Situation calculus:

- conventions for describing actions and change in FOL
- can formulate planning as inference on a situation calculus KB

Inference in first-order logic

Chapter 9

Outline

- ♢ Reducing first-order inference to propositional inference
- ♢ Unification
- ♢ Generalized Modus Ponens
- ♢ Forward and backward chaining
- ♢ Logic programming
- ♢ Resolution

A brief history of reasoning

450B.C.	Stoics	propositional logic, inference (maybe)
322B.C.	Aristotle	"syllogisms" (inference rules), quantifiers
1565	Cardano	probability theory (propositional logic + uncertainty)
1847	Boole	propositional logic (again)
1879	Frege	first-order logic
1922	Wittgenstein	proof by truth tables
1930	Gödel	$\exists$ complete algorithm for FOL
1930	Herbrand	complete algorithm for FOL (reduce to propositional)
1931	Gödel	$\neg\exists$ complete algorithm for arithmetic
1960	Davis/Putnam	"practical" algorithm for propositional logic
1965	Robinson	"practical" algorithm for FOL—resolution

Universal instantiation (UI)

Every instantiation of a universally quantified sentence is entailed by it:

$$\frac{\forall v \ \ \alpha}{\textsc{Subst}(\{v/g\}, \alpha)}$$

for any variable v and ground term g

E.g., $\forall x \ \ King(x) \wedge Greedy(x) \ \Rightarrow \ Evil(x)$ yields

$$King(John) \wedge Greedy(John) \ \Rightarrow \ Evil(John)$$
$$King(Richard) \wedge Greedy(Richard) \ \Rightarrow \ Evil(Richard)$$
$$King(Father(John)) \wedge Greedy(Father(John)) \ \Rightarrow \ Evil(Father(John))$$
$$\vdots$$

Existential instantiation (EI)

For any sentence α, variable v, and constant symbol k
that does not appear elsewhere in the knowledge base:

$$\frac{\exists v \;\; \alpha}{\text{SUBST}(\{v/k\}, \alpha)}$$

E.g., $\exists x \;\; Crown(x) \wedge OnHead(x, John)$ yields

$$Crown(C_1) \wedge OnHead(C_1, John)$$

provided C_1 is a new constant symbol, called a Skolem constant

Another example: from $\exists x \;\; d(x^y)/dy = x^y$ we obtain

$$d(e^y)/dy = e^y$$

provided e is a new constant symbol

Existential instantiation contd.

UI can be applied several times to **add** new sentences;
the new KB is logically equivalent to the old

EI can be applied once to **replace** the existential sentence;
the new KB is **not** equivalent to the old,
but is satisfiable iff the old KB was satisfiable

Question: How can we use instantiations for inference in FOL?

Reduction to propositional inference

Suppose the KB contains just the following:

$$\forall x \;\; King(x) \wedge Greedy(x) \;\Rightarrow\; Evil(x)$$
$$King(John)$$
$$Greedy(John)$$
$$Brother(Richard, John)$$

Instantiating the universal sentence in **all possible** ways, we have

$$King(John) \wedge Greedy(John) \;\Rightarrow\; Evil(John)$$
$$King(Richard) \wedge Greedy(Richard) \;\Rightarrow\; Evil(Richard)$$
$$King(John)$$
$$Greedy(John)$$
$$Brother(Richard, John)$$

The new KB is propositionalized: proposition symbols are

$$King(John),\; Greedy(John),\; Evil(John), King(Richard)$$ etc.

Reduction contd.

Claim: a ground sentence* is entailed by new KB iff entailed by original KB

Claim: every FOL KB can be propositionalized so as to preserve entailment

Idea: propositionalize KB and query, apply resolution, return result

Problem: with function symbols, there are infinitely many ground terms,
e.g., $Father(Father(Father(John)))$

Theorem: Herbrand (1930). If a sentence α is entailed by an FOL KB,
it is entailed by a **finite** subset of the propositional KB

Idea: For $n = 0$ to ∞ do
create a propositional KB by instantiating with depth-n terms
see if α is entailed by this KB

Via propositionalization, any entailed sentence can be proved: complete!
What if not entailed? We cannot tell if it is entailed or not...

Problems with propositionalization

Propositionalization seems to generate lots of irrelevant sentences.
E.g., from

$$\forall x \;\; King(x) \wedge Greedy(x) \;\Rightarrow\; Evil(x)$$
$$King(John)$$
$$\forall y \;\; Greedy(y)$$
$$Brother(Richard, John)$$

it seems obvious that $Evil(John)$, but propositionalization produces lots of facts such as $Greedy(Richard)$ that are irrelevant

With p k-ary predicates and n constants, there are $p \cdot n^k$ instantiations

With function symbols, it gets nuch much worse!

A first-order inference rule

Indeed, the inference of $Evil(John)$ from the sentences

$$\forall x \;\; King(x) \wedge Greedy(x) \;\Rightarrow\; Evil(x)$$
$$King(John)$$
$$Greedy(John)$$

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$$\forall x \;\; King(x) \wedge Greedy(x) \;\Rightarrow\; Evil(x)$$
$$King(John)$$
$$Greedy(John)$$

$\{x/John\}$ solves the query $Evil(x)$.

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$$King(John)$$
$$\forall y \;\; Greedy(y) \; .$$

$$\{x/John, y/John\}$$

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$$\{x/John, y/John\}$$

Generalized Modus Ponens (GMP)

$$\frac{p_1', \;\; p_2', \;\; \ldots, \;\; p_n', \;\; (p_1 \wedge p_2 \wedge \ldots \wedge p_n \Rightarrow q)}{q\theta}$$

where $p_i'\theta = p_i\theta$ for all i

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where $p_i'\theta = p_i\theta$ for all i

p_1' is $King(John)$ $\quad$ p_1 is $King(x)$

p_2' is $Greedy(y)$ $\quad$ p_2 is $Greedy(x)$

q is $Evil(x)$

A first-order inference rule

Indeed, the inference of $Evil(John)$ from the sentences

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$$King(John)$$
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p_2' is $Greedy(y)$

p_1 is $King(x)$
p_2 is $Greedy(x)$
q is $Evil(x)$

θ is $\{x/John, y/John\}$
$q\theta$ is $Evil(John)$

Unification

$Knows(John, Jane)$

$\forall y \; Knows(y, Bill)$

$\forall y \; Knows(y, Mother(y))$

$\forall x \; Knows(x, Elizabeth)$

$AskVars(Knows(John, x))$: whom does John know?

Unification

We can get the inference immediately if we can find a substitution θ such that $King(x)$ and $Greedy(x)$ match $King(John)$ and $Greedy(y)$

$\theta = \{x/John, y/John\}$ works

$\textsc{Unify}(\alpha, \beta) = \theta$ if $\alpha\theta = \beta\theta$

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	
$Knows(John, x)$	$Knows(y, OJ)$	
$Knows(John, x)$	$Knows(y, Mother(y))$	
$Knows(John, x)$	$Knows(x, OJ)$	

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p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	$\{x/Jane\}$
$Knows(John, x)$	$Knows(y, OJ)$	
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$Knows(John, x)$	$Knows(y, OJ)$	$\{x/OJ, y/John\}$
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$Knows(John, x)$	$Knows(y, OJ)$	$\{x/OJ, y/John\}$
$Knows(John, x)$	$Knows(y, Mother(y))$	$\{y/John, x/Mother(John)\}$
$Knows(John, x)$	$Knows(x, OJ)$	

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$Knows(John, x)$	$Knows(y, OJ)$	$\{x/OJ, y/John\}$
$Knows(John, x)$	$Knows(y, Mother(y))$	$\{y/John, x/Mother(John)\}$
$Knows(John, x)$	$Knows(x, OJ)$	$fail$

Standardizing apart eliminates overlap of variables, e.g., $Knows(z_{17}, OJ)$

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θ is $\{x/John, y/John\}$ $\quad$ q is $Evil(x)$
$q\theta$ is $Evil(John)$

GMP used with KB of definite clauses (**exactly** one positive literal)
All variables assumed universally quantified

Example knowledge base

The law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.

Prove that Col. West is a criminal

Example knowledge base contd.

. . . it is a crime for an American to sell weapons to hostile nations:

Example knowledge base contd.

. . . it is a crime for an American to sell weapons to hostile nations:

$$American(x) \wedge Weapon(y) \wedge Sells(x, y, z) \wedge Hostile(z) \Rightarrow Criminal(x)$$

Nono . . . has some missiles

Example knowledge base contd.

. . . it is a crime for an American to sell weapons to hostile nations:

$American(x) \land Weapon(y) \land Sells(x, y, z) \land Hostile(z) \Rightarrow Criminal(x)$

Nono . . . has some missiles, i.e., $\exists x\ Owns(Nono, x) \land Missile(x)$:

$Owns(Nono, M_1)$ and $Missile(M_1)$

. . . all of its missiles were sold to it by Colonel West

Example knowledge base contd.

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$\forall x\ \ Missile(x) \wedge Owns(Nono, x) \ \Rightarrow\ Sells(West, x, Nono)$

Missiles are weapons:

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Missiles are weapons:

$Missile(x) \Rightarrow Weapon(x)$

An enemy of America counts as "hostile":

Example knowledge base contd.

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Missiles are weapons:

$Missile(x) \Rightarrow Weapon(x)$

An enemy of America counts as "hostile":

$Enemy(x, America) \ \Rightarrow\ Hostile(x)$

West, who is American . . .

$American(West)$

The country Nono, an enemy of America . . .

$Enemy(Nono, America)$