Jacobian

Jacobian

- Defines dynamic relationship between two different representations of a system.
- If we have a 2-link robotic arm, there are two obvious ways to describe its current position. ?
- The Jacobian for this system relates how movement of the elements of q causes movement of the elements of x.
- Formally, a Jacobian is a set of partial differential equations:

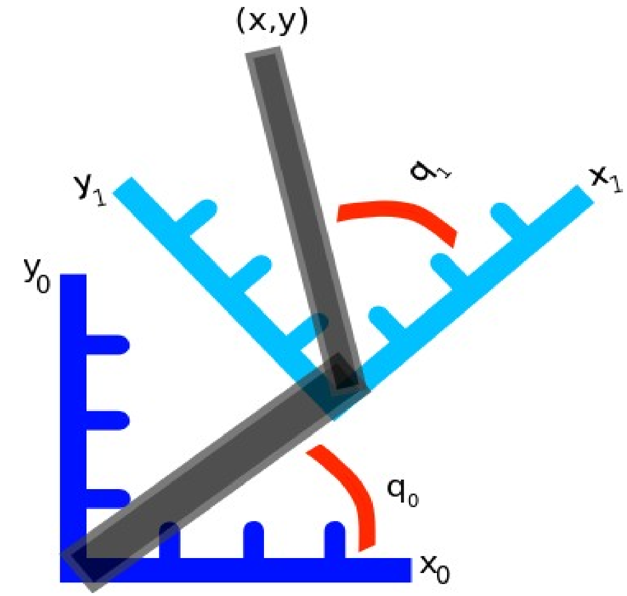
$$\mathbf{J} = \frac{\partial \mathbf{x}}{\partial \mathbf{q}}$$

- With a bit of manipulation we can get a neat result

$$\mathbf{J} = \frac{\partial \mathbf{x}}{\partial t} \frac{\partial t}{\partial \mathbf{q}} \rightarrow \frac{\partial \mathbf{x}}{\partial \mathbf{t}} = \mathbf{J} \frac{\partial \mathbf{q}}{\partial t} \qquad \dot{\mathbf{x}} = \mathbf{J}\,\dot{\mathbf{q}}$$

- Why is this important?
- BUT what we'd like is to plan our trajectory in terms of end-effector position (and possibly orientation), generating control signals in terms of forces to apply in (x,y,z) space.

Building the Jacobian

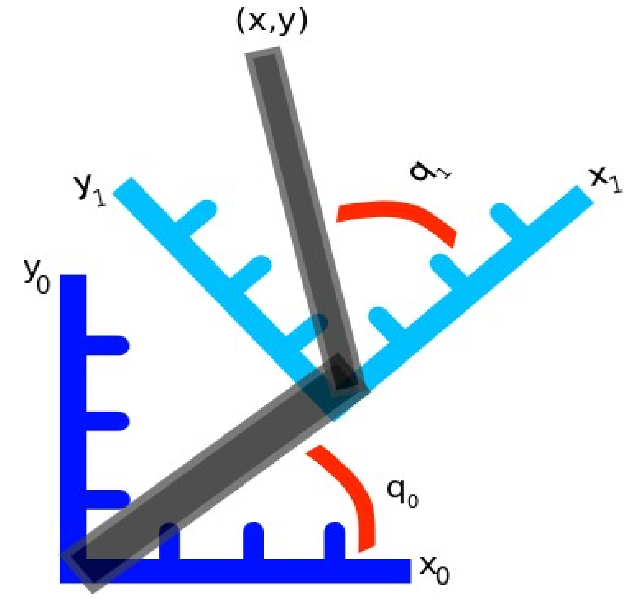


- The forward transformation matrix
- Allow a given point to be transformed between different reference frames.
- The rotation part of this matrix is straight-forward to define

$$\mathbf{R}_0^1 = \begin{bmatrix} cos(q_0) & -sin(q_0) & 0 \\ sin(q_0) & cos(q_0) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \mathbf{D}_0^1 = \begin{bmatrix} L_0 cos(q_0) \\ L_0 sin(q_0) \\ 0 \end{bmatrix} . \mathbf{x} = \begin{bmatrix} L_1 cos(q_1) \\ L_1 sin(q_1) \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{T}_0^1 \, \mathbf{x} = \begin{bmatrix} cos(q_0) & -sin(q_0) & 0 & L_0 cos(q_0) \\ sin(q_0) & cos(q_0) & 0 & L_0 sin(q_0) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L_1 cos(q_1) \\ L_1 sin(q_1) \\ 0 \\ 1 \end{bmatrix},$$

Building the Jacobian



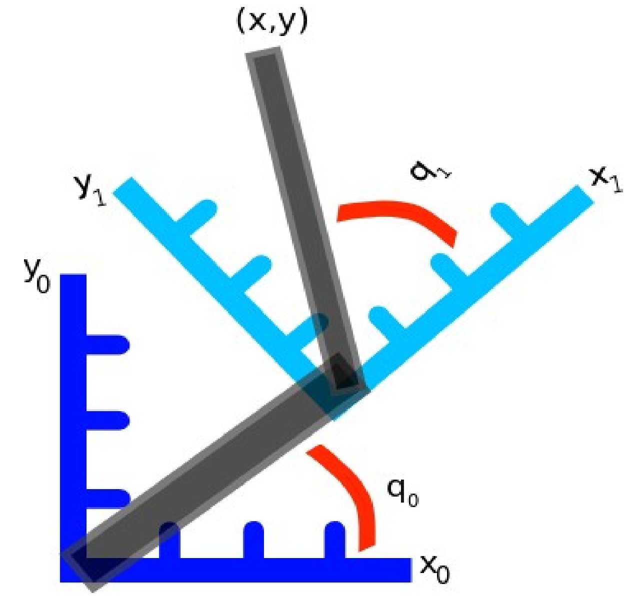
- Position of end-effector

$$\begin{bmatrix} L_0 cos(q_0) + L_1 cos(q_0 + q_1) \\ L_0 sin(q_0) + L_1 sin(q_0 + q_1) \\ 0 \end{bmatrix},$$

- Accounting for orientation:

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q_0 + q_1 \end{bmatrix},$$

Building the Jacobian

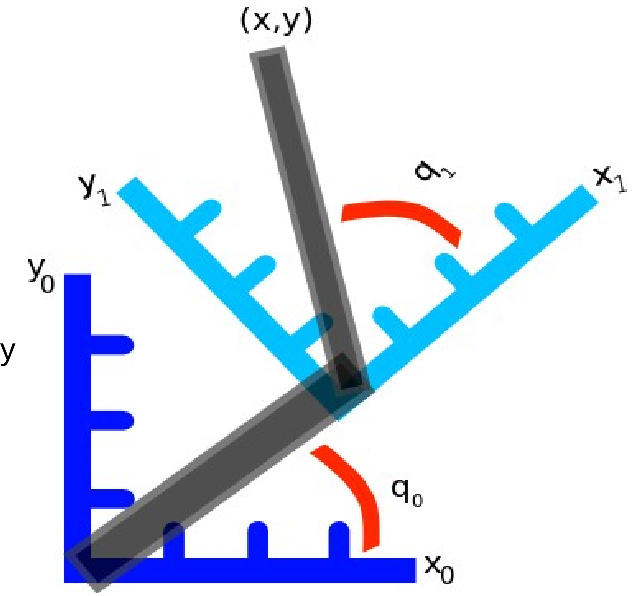


$$\begin{bmatrix} L_0 cos(q_0) + L_1 cos(q_0 + q_1) \\ L_0 sin(q_0) + L_1 sin(q_0 + q_1) \\ 0 \end{bmatrix}, \qquad \mathbf{J} = \frac{\partial \mathbf{x}}{\partial \mathbf{q}}$$

$$\mathbf{J}_v(\mathbf{q}) = \begin{bmatrix} \frac{\partial x}{\partial q_0} & \frac{\partial x}{\partial q_1} \\ \frac{\partial y}{\partial q_0} & \frac{\partial y}{\partial q_1} \\ \frac{\partial z}{\partial q_0} & \frac{\partial z}{\partial q_1} \end{bmatrix} = \begin{bmatrix} -L_0 sin(q_0) - L_1 sin(q_0 + q_1) & -L_1 sin(q_0 + q_1) \\ L_0 cos(q_0) + L_1 cos(q_0 + q_1) & L_1 cos(q_0 + q_1) \\ 0 & 0 \end{bmatrix}.$$

$$\mathbf{J}_\omega(\mathbf{q}) = \begin{bmatrix} \frac{\partial \omega_x}{\partial q_0} & \frac{\partial \omega_x}{\partial q_1} \\ \frac{\partial \omega_y}{\partial q_0} & \frac{\partial \omega_y}{\partial q_1} \\ \frac{\partial \omega_z}{\partial q_0} & \frac{\partial \omega_z}{\partial q_1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Building the Jacobian

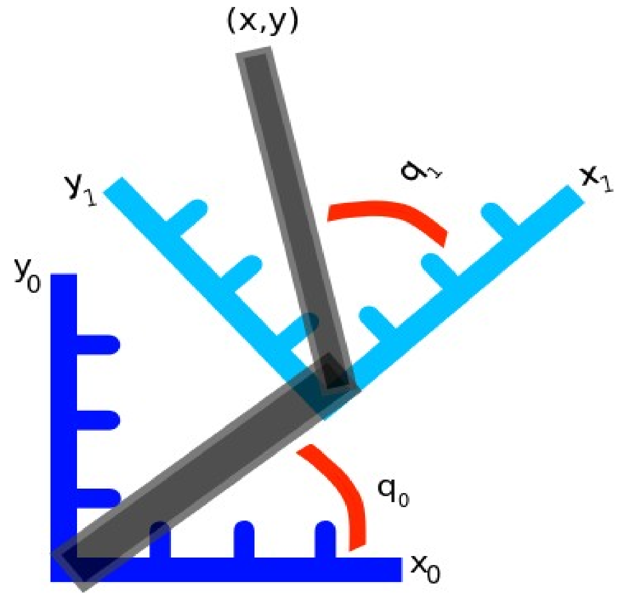


- This means that the z position, and rotations w_x and w_y are not controllable. These rows of zeros in the Jacobian are referred to as its `null space'.

$$\dot{\mathrm{x}} = \mathrm{J}_{ee}(\mathrm{q})\,\dot{\mathrm{q}}.$$

$$\mathbf{J}_{ee}(\mathbf{q}) = \begin{bmatrix} \mathbf{J}_v(\mathbf{q}) \\ \mathbf{J}_\omega(\mathbf{q}) \end{bmatrix} = \begin{bmatrix} -L_0 sin(q_0) - L_1 sin(q_0+q_1) & -L_1 sin(q_0+q_1) \\ L_0 cos(q_0) + L_1 cos(q_0+q_1) & L_1 cos(q_0+q_1) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Using the Jacobian, example 1



- Given known joint angle velocities,
  - find end effector linear and angular velocities

$$\mathbf{q} = \begin{bmatrix} \frac{\pi}{4} \\ \frac{3\pi}{8} \end{bmatrix} \quad \dot{\mathbf{q}} = \begin{bmatrix} \frac{\pi}{10} \\ \frac{\pi}{10} \end{bmatrix} \quad L_i = 1$$

$$\dot{\mathbf{x}} = \mathbf{J}_{ee}(\mathbf{q})\, \dot{\mathbf{q}},$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -sin(\frac{\pi}{4}) - sin(\frac{\pi}{4} + \frac{3\pi}{8}) & -sin(\frac{\pi}{4} + \frac{3\pi}{8}) \\ cos(\frac{\pi}{4}) + cos(\frac{\pi}{4} + \frac{3\pi}{8}) & cos(\frac{\pi}{4} + \frac{3\pi}{8}) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\pi}{10} \\ \frac{\pi}{10} \end{bmatrix},$$

$$\dot{\mathbf{x}} = \left[-0.8026, -0.01830, 0, 0, 0, \frac{\pi}{5}\right]^T.$$

Energy equivalence and Jacobians

- Let the joint angle positions be denoted $q = [q_0, q_1]^T$ and end-effector position be denoted $x = [x, y, 0]^T$.
- Work is the application of force over a distance

$$W = Fs \qquad \mathbf{W} = \int \mathbf{F}^T \mathbf{v} \, dt,$$

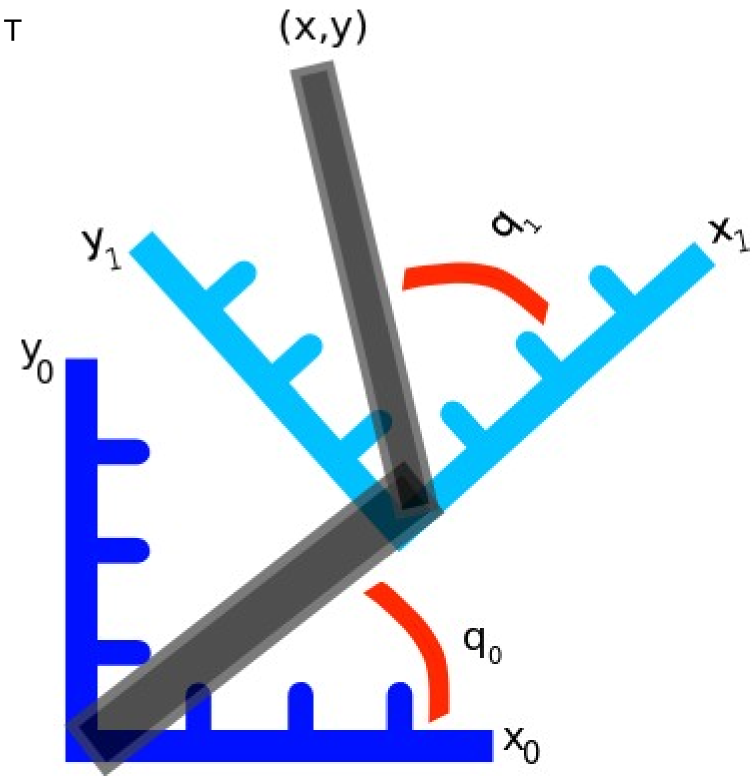
- Power is the amount of energy consumed per unit time

$$\mathbf{P} = \frac{d}{dt}\mathbf{W},$$

- Substitute work into the equation for power gives

$$\mathbf{P} = \frac{d}{dt}\mathbf{F}^T \mathbf{v}.$$

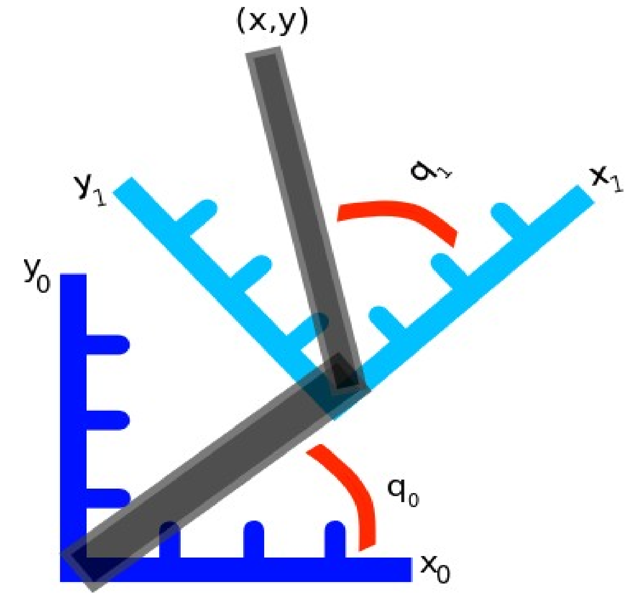
- Rewriting the above in terms of end-effector space and joint-space give:



$$\mathbf{P} = \mathbf{F}_{\mathbf{x}}^T \dot{\mathbf{x}}, \qquad \mathbf{F}_{q_{hand}}^T \dot{\mathbf{q}} = \mathbf{F}_{\mathbf{x}}^T \dot{\mathbf{x}},$$
$$\mathbf{P} = \mathbf{F}_{\mathbf{q}}^T \dot{\mathbf{q}}, \qquad \mathbf{F}_{q_{hand}}^T \dot{\mathbf{q}} = \mathbf{F}_{\mathbf{x}}^T \mathbf{J}_{ee}(\mathbf{q}) \, \dot{\mathbf{q}},$$

$$\mathbf{F}_{q_{hand}} = \mathbf{J}_{ee}^T(\mathbf{q}) \mathbf{F}_{\mathbf{x}}.$$

Building Jacobian



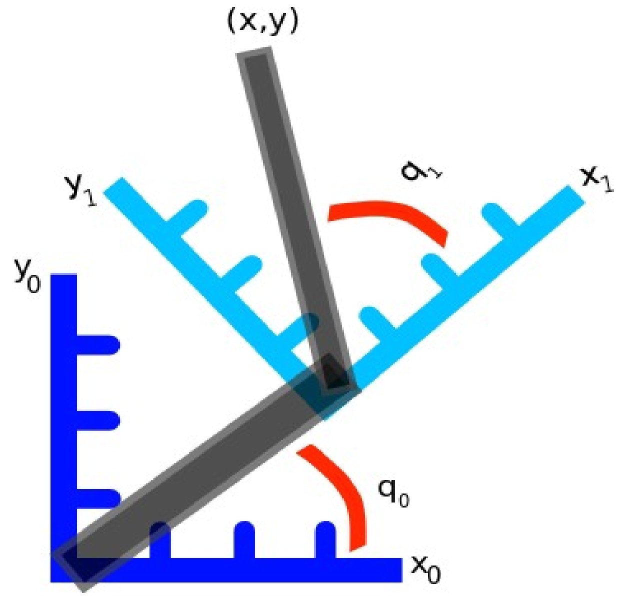
- Looking at the variables that can be affected it can be seen that given any two of x, y, w_z the third can be calculated.

$$\mathbf{F_x} = \begin{bmatrix} f_x \\ f_y \end{bmatrix},$$

$$\mathbf{J}_{ee}(\mathbf{q}) = \begin{bmatrix} -L_0 sin(q_0) - L_1 sin(q_0 + q_1) & -L_1 sin(q_0 + q_1) \\ L_0 cos(q_0) + L_1 cos(q_0 + q_1) & L_1 cos(q_0 + q_1) \end{bmatrix}.$$

$$\mathbf{F}_{q_{hand}} = \mathbf{J}_{ee}^T(\mathbf{q})\mathbf{F_x}.$$

Using the Jacobian, example 2



- The goal is to calculate the torques required to get the end-effector to move as desired. The controlled variables will be forces along the x and y axes.
- Let the desired (x,y) forces be

$$\mathbf{F_x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{F_q} = \mathbf{J}_{ee}^T(\mathbf{q})\mathbf{F_x},$$

$$\mathbf{J}_{ee}(\mathbf{q}) = \begin{bmatrix} -L_0 sin(q_0) - L_1 sin(q_0+q_1) & -L_1 sin(q_0+q_1) \\ L_0 cos(q_0) + L_1 cos(q_0+q_1) & L_1 cos(q_0+q_1) \end{bmatrix}.$$

$$\mathbf{q} = \begin{bmatrix} \frac{\pi}{4} \\ \frac{3\pi}{8} \end{bmatrix} \quad \dot{\mathbf{q}} = \begin{bmatrix} \frac{\pi}{10} \\ \frac{\pi}{10} \end{bmatrix}$$

$$\mathbf{F_q} = \begin{bmatrix} -sin(\frac{\pi}{4}) - sin(\frac{\pi}{4} + \frac{3\pi}{8}) & cos(\frac{\pi}{4}) + cos(\frac{\pi}{4} + \frac{3\pi}{8}) \\ -sin(\frac{\pi}{4} + \frac{3\pi}{8}) & cos(\frac{\pi}{4} + \frac{3\pi}{8}) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\mathbf{F_q} = \begin{bmatrix} -1.3066 \\ -1.3066 \end{bmatrix}.$$