# Heapsort

# Why study Heapsort?

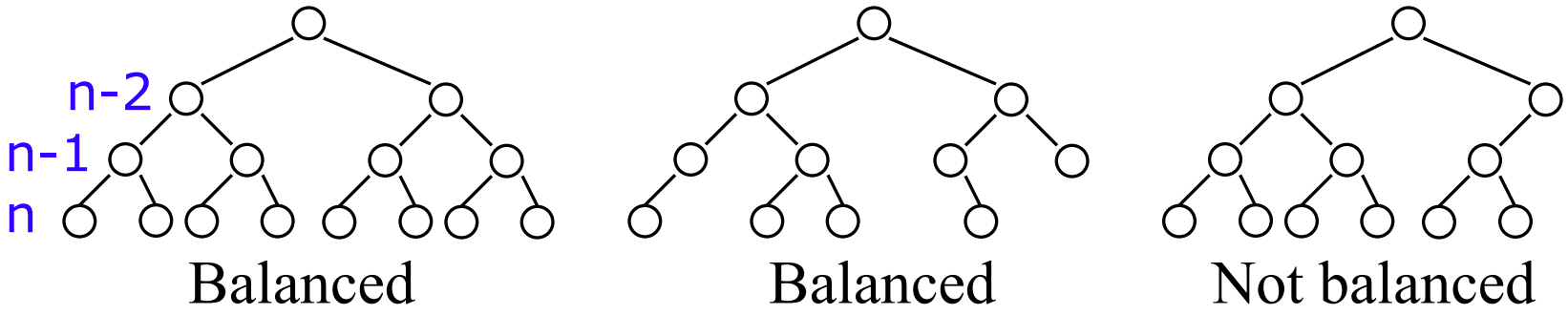
- It is a well-known, traditional sorting algorithm you will be expected to know
- Heapsort is *always* O(n log n)
  - Quicksort is usually O(n log n) but in the worst case slows to $O(n^2)$
  - Quicksort is generally faster, but Heapsort is better in time-critical applications
- Heapsort is a *really cool* algorithm!

# What is a "heap"?

- Definitions of heap:
  1. A large area of memory from which the programmer can allocate blocks as needed, and deallocate them (or allow them to be garbage collected) when no longer needed
  2. A balanced, left-justified binary tree in which no node has a value greater than the value in its parent
- These two definitions have little in common
- Heapsort uses the second definition

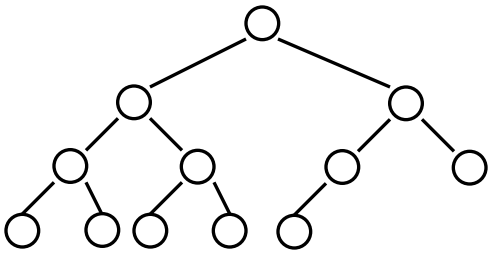
# Balanced binary trees

- Recall:
  - The depth of a node is its distance from the root
  - The depth of a tree is the depth of the deepest node
- A binary tree of depth `n` is balanced if all the nodes at depths `0` through `n-2` have two children

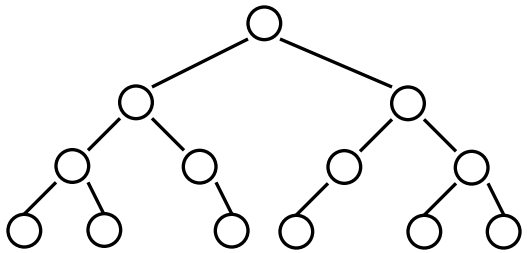


# Left-justified binary trees

- A balanced binary tree of depth $n$ is left-justified if:
  - it has $2^n$ nodes at depth $n$ (the tree is "full"), or
  - it has $2^k$ nodes at depth $k$, for all $k < n$, *and* all the leaves at depth $n$ are as far left as possible



Left-justified



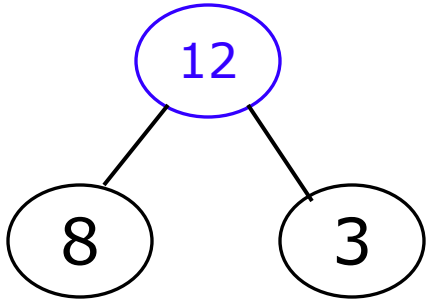
Not left-justified

# Plan of attack

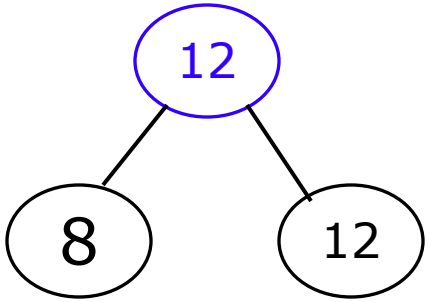
- First, we will learn how to turn a binary tree into a heap
- Next, we will learn how to turn a binary tree *back* into a heap after it has been changed in a certain way
- Finally (this is the cool part) we will see how to use these ideas to sort an *array*

# The heap property

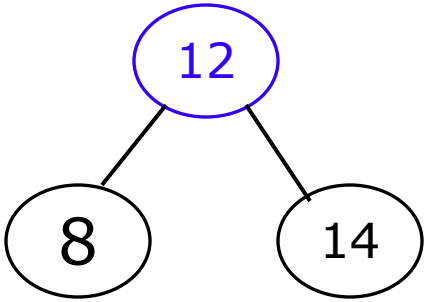
- A node has the heap property if the value in the node is as large as or larger than the values in its children




Blue node has heap property




Blue node has heap property

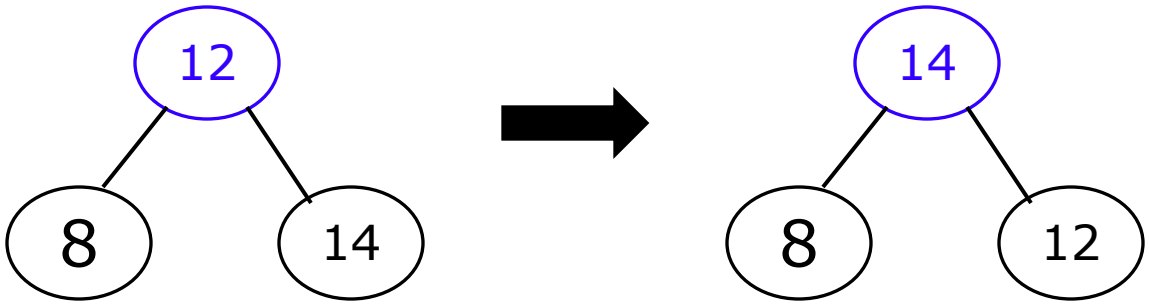



Blue node does not have heap property

- All leaf nodes automatically have the heap property
- A binary tree is a heap if *all* nodes in it have the heap property

# siftUp

- Given a node that does not have the heap property, you can give it the heap property by exchanging its value with the value of the larger child



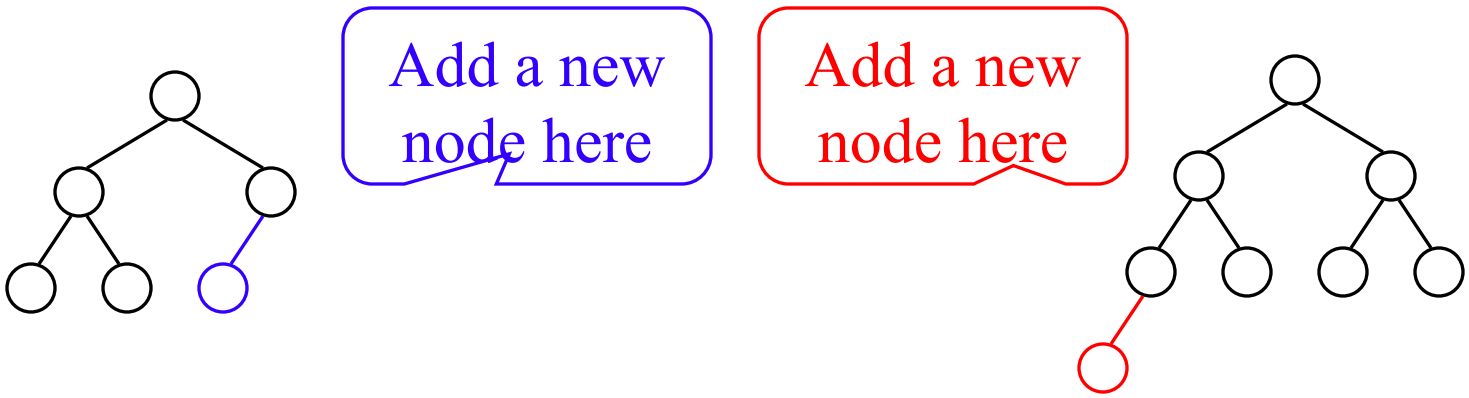

Blue node does not have heap property

Blue node has heap property

- This is sometimes called sifting up
- Notice that the child may have *lost* the heap property

# Constructing a heap I

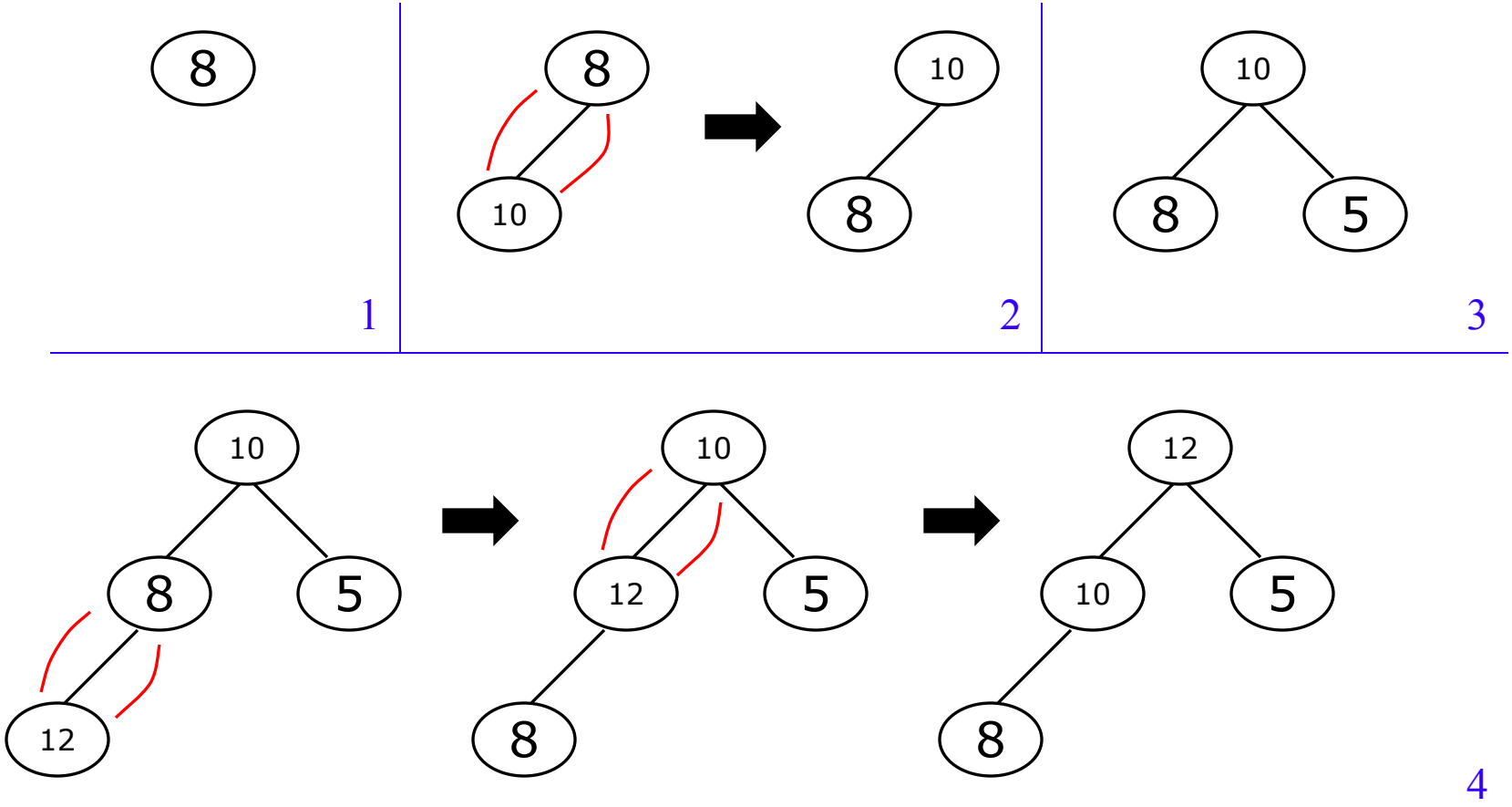
- A tree consisting of a single node is automatically a heap
- We construct a heap by adding nodes one at a time:
  - Add the node just to the right of the rightmost node in the deepest level
  - If the deepest level is full, start a new level
- Examples:



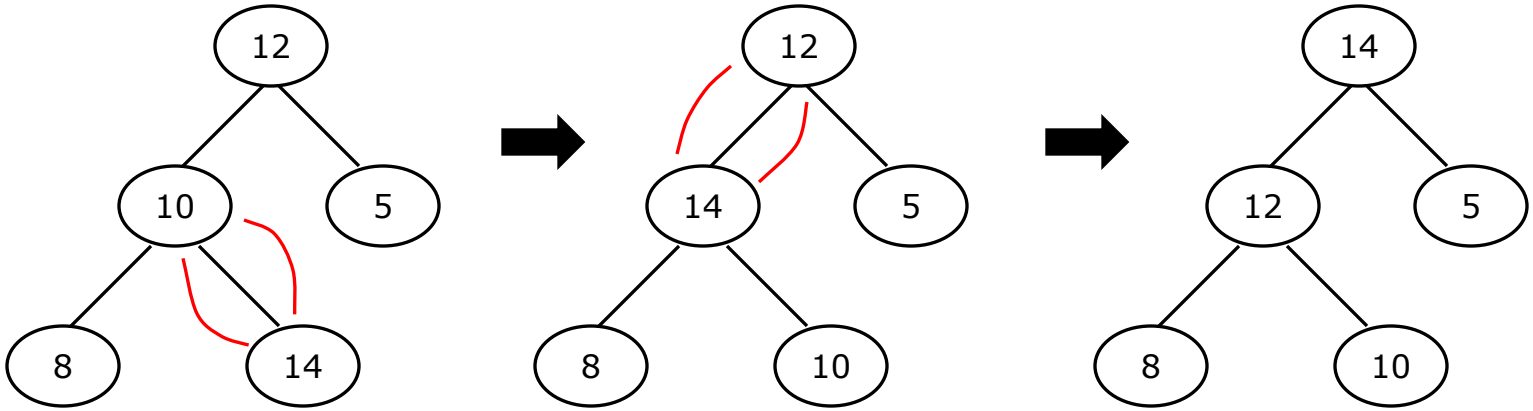
# Constructing a heap II

- Each time we add a node, we may destroy the heap property of its parent node
- To fix this, we sift up
- But each time we sift up, the value of the topmost node in the sift may increase, and this may destroy the heap property of *its* parent node
- We repeat the sifting up process, moving up in the tree, until either
  - We reach nodes whose values don't need to be swapped (because the parent is *still* larger than both children), or
  - We reach the root

# Constructing a heap III



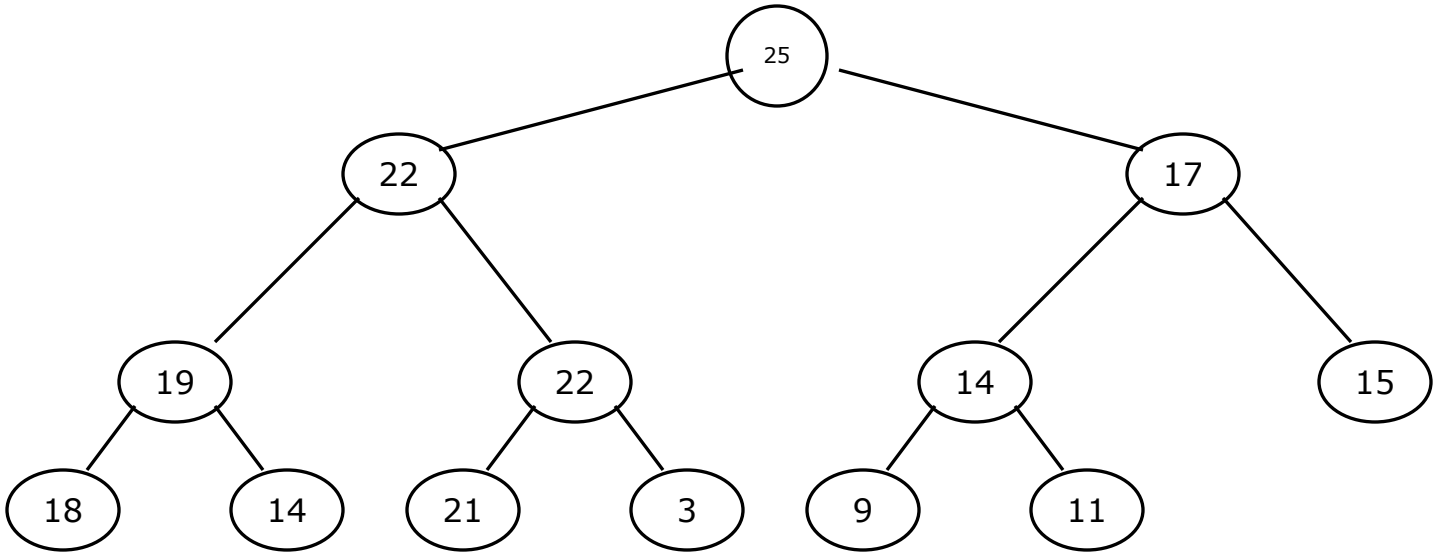
# Other children are not affected




- The node containing 8 is not affected because its parent gets larger, not smaller
- The node containing 5 is not affected because its parent gets larger, not smaller
- The node containing 8 is still not affected because, although its parent got smaller, its parent is still greater than it was originally

# A sample heap

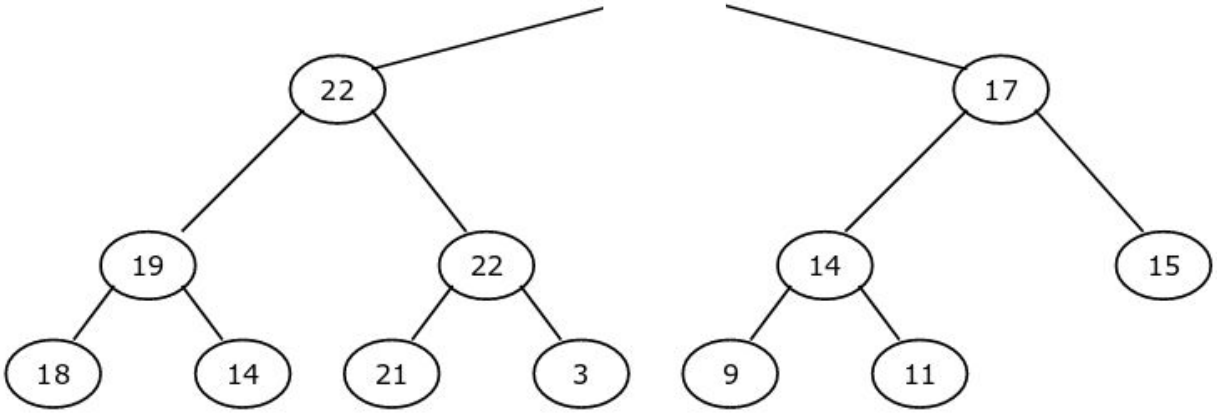
- Here's a sample binary tree after it has been heapified




- Notice that heapified does *not* mean sorted
- Heapifying does *not* change the shape of the binary tree; this binary tree is balanced and left-justified because it started out that way

# Removing the root

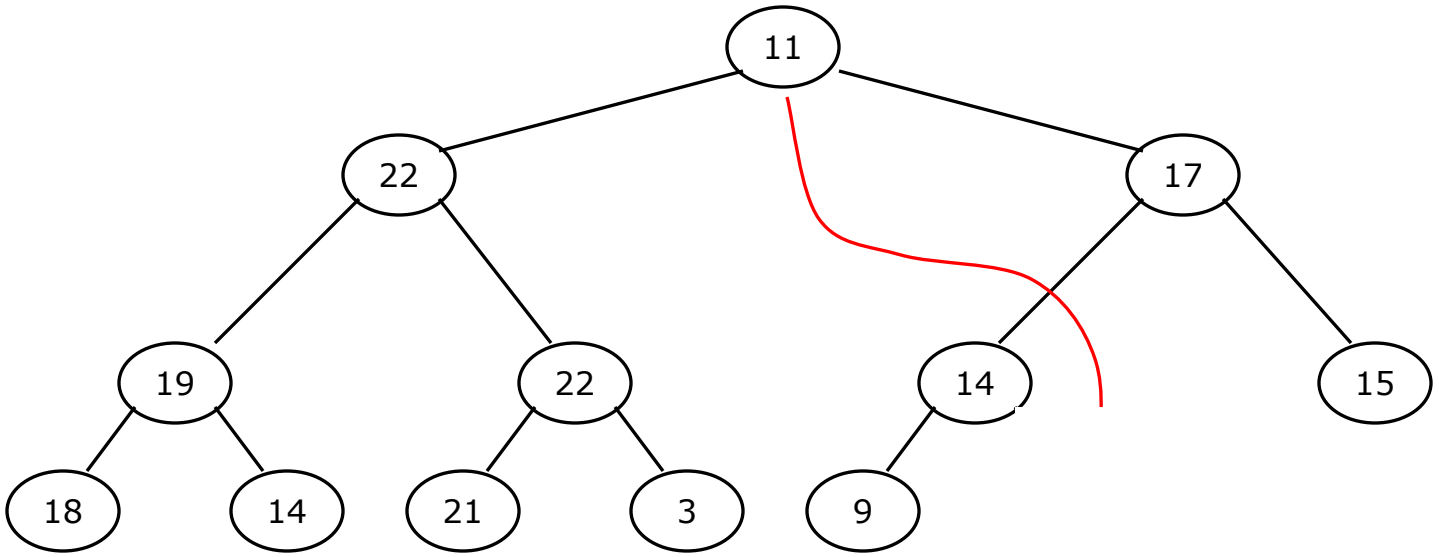
- Notice that the largest number is now in the root
- Suppose we *discard* the root:




We need to "promote" some other node to be the new root

# Removing the root (animated)

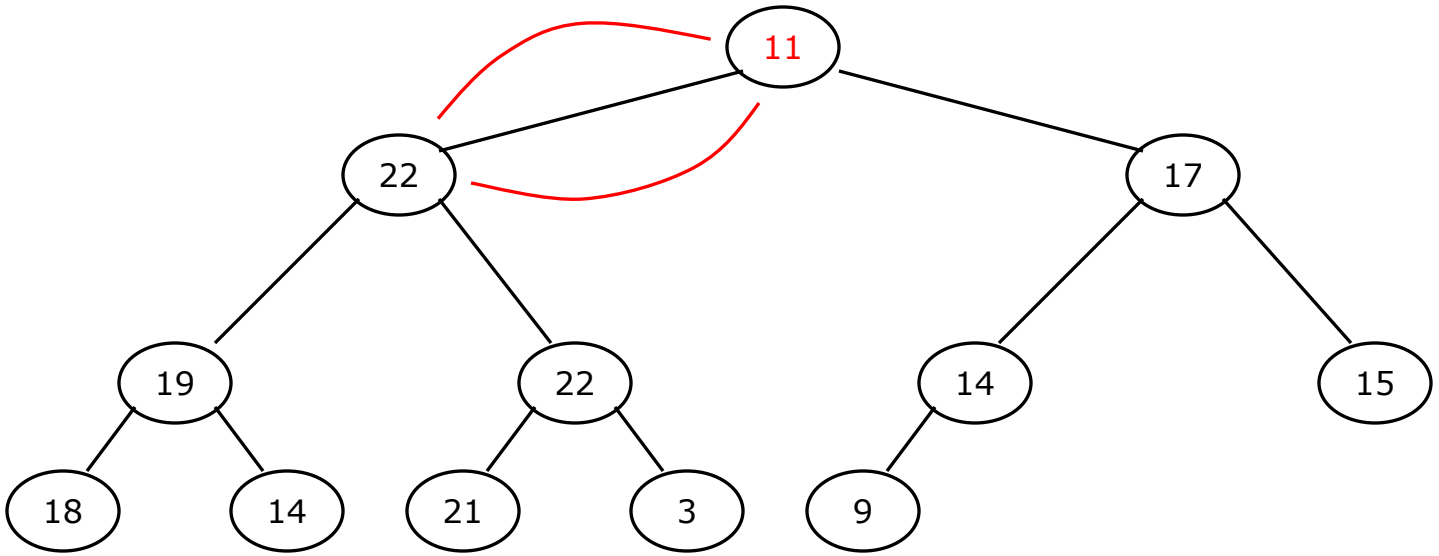
- Here's our "rootless" binary tree:




- How can we fix the binary tree so it is once again *balanced and left-justified?*
- Solution: remove the rightmost leaf at the deepest level and use it for the new root

# The reHeap method I

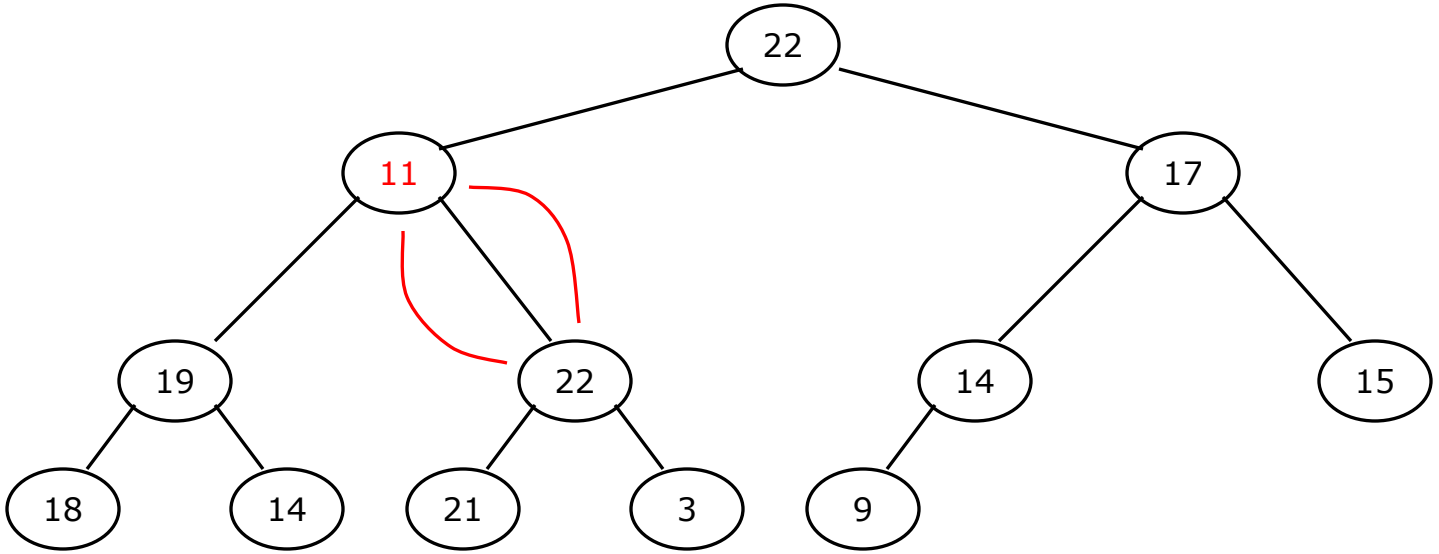
- Our tree is balanced and left-justified, but no longer a heap
- However, *only the root* lacks the heap property




- We can siftUp() the root
- After doing this, one and only one of its children may have lost the heap property

# The reHeap method II

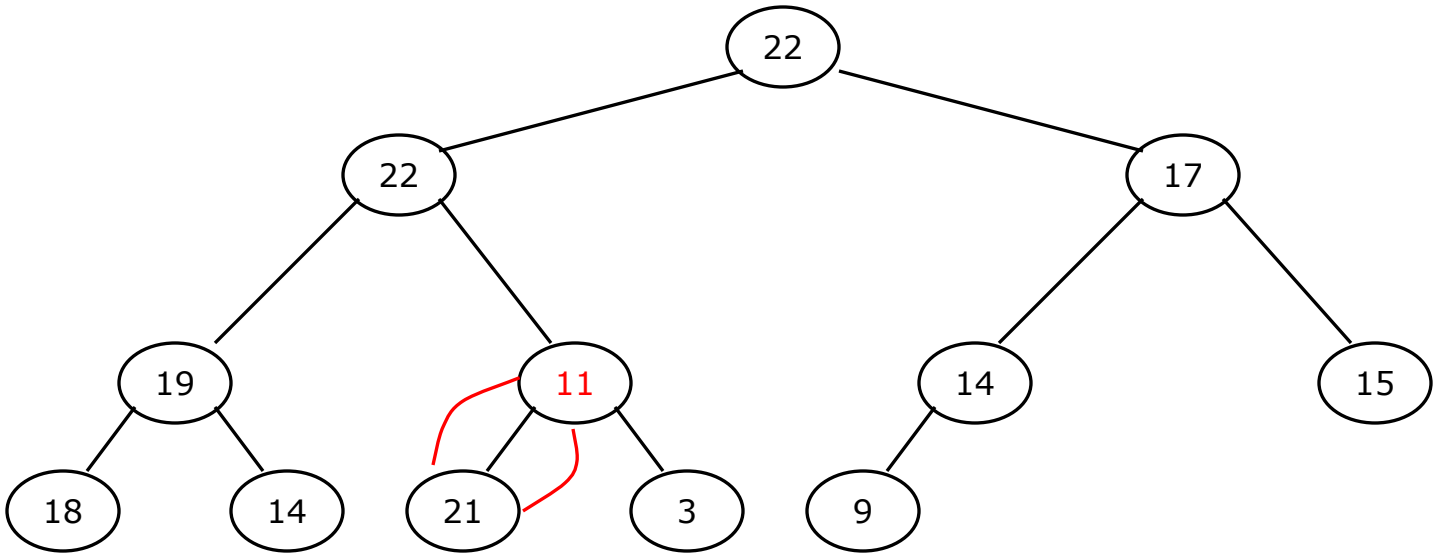
- Now the left child of the root (still the number 11) lacks the heap property




- We can siftUp() this node
- After doing this, one and only one of its children may have lost the heap property

# The reHeap method III

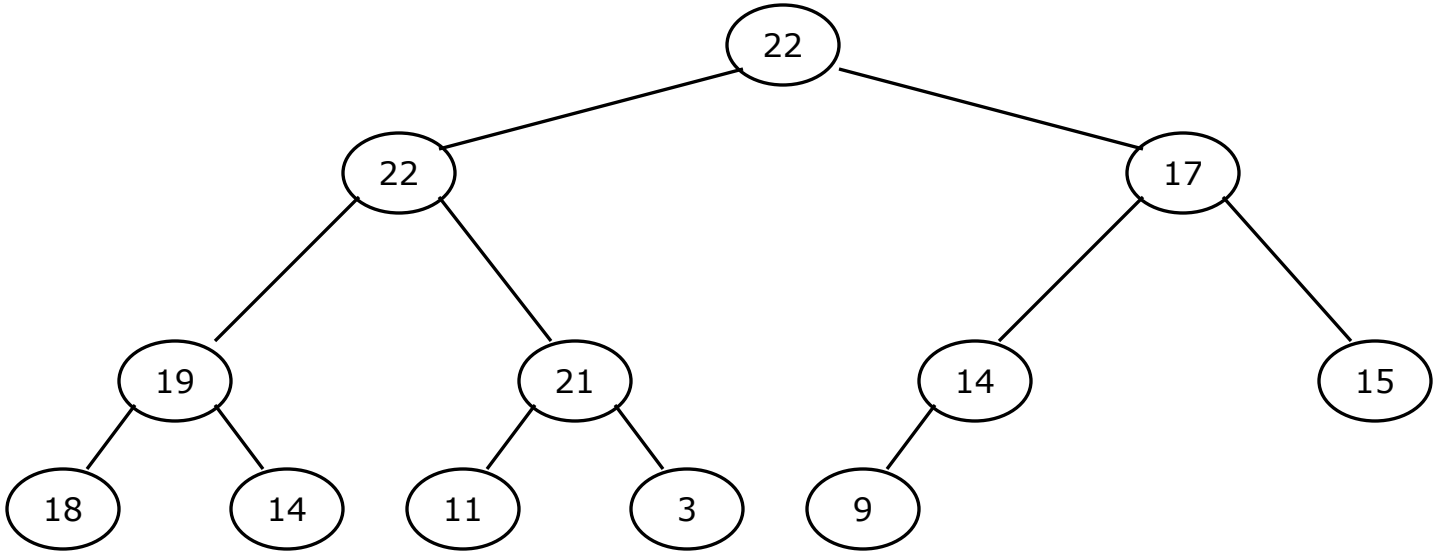
- Now the right child of the left child of the root (still the number 11) lacks the heap property:




- We can siftUp() this node
- After doing this, one and only one of its children may have lost the heap property —but it doesn't, because it's a leaf

# The reHeap method IV

- Our tree is once again a heap, because every node in it has the heap property



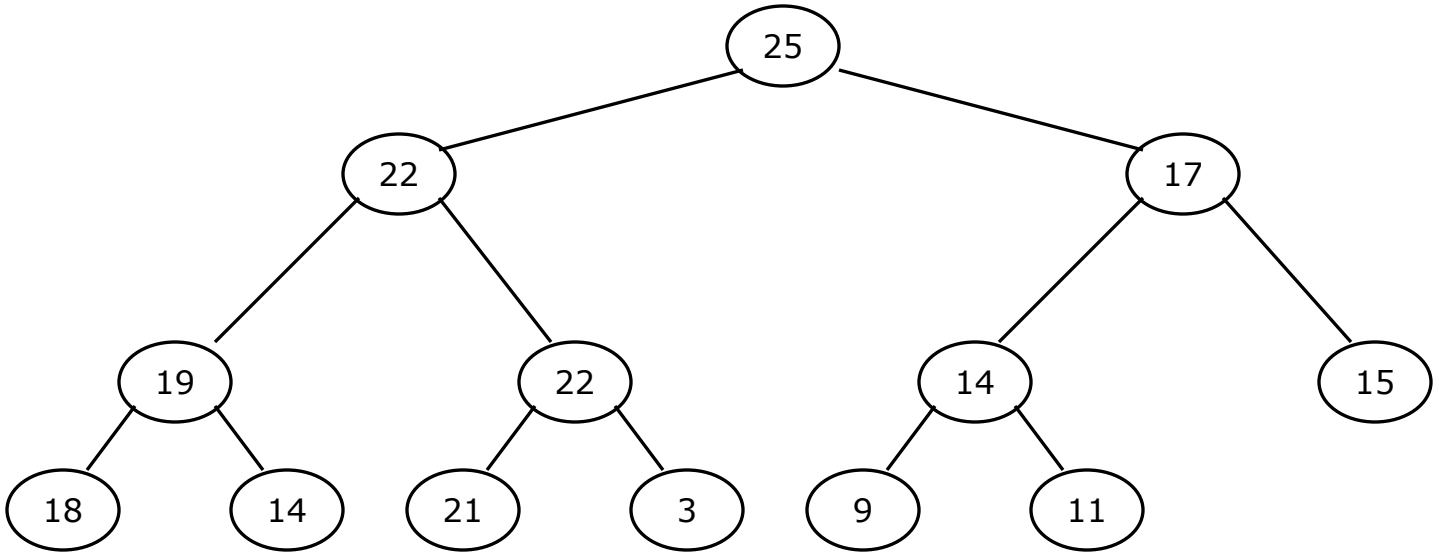

- Once again, the largest (or *a* largest) value is in the root
- We can repeat this process until the tree becomes empty
- This produces a sequence of values in order largest to smallest

# Sorting

- What do heaps have to do with sorting an array?
- Here's the neat part:
  - Because the binary tree is *balanced* and *left justified,* it can be represented as an array
    - ***Danger:*** This representation works well ***only*** with ***balanced***, ***left-justified*** binary trees
  - All our operations on binary trees can be represented as operations on *arrays*
  - To sort:

```
heapify the array;
while the array isn't empty {
    remove and replace the root;
    reheap the new root node;
}
```

# Mapping into an array

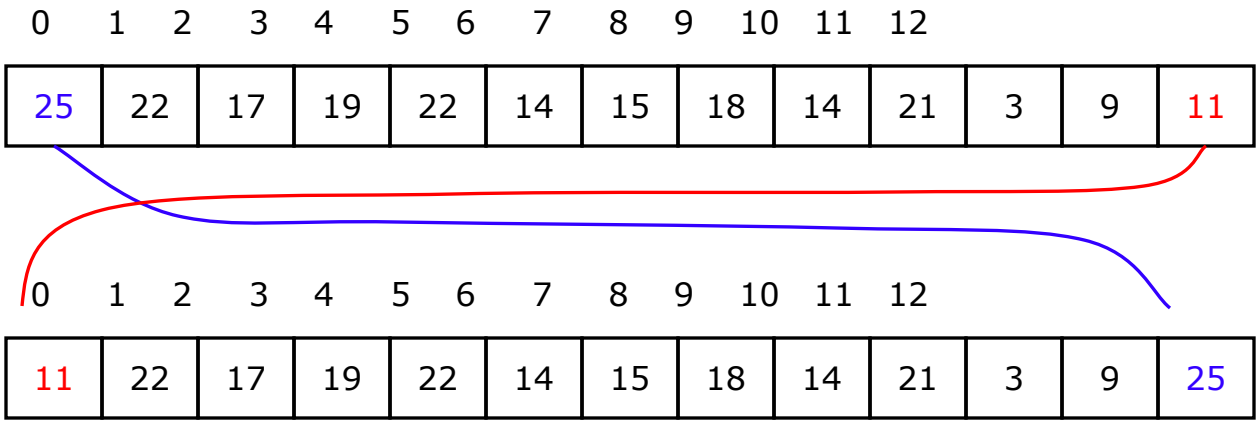



| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 25 | 22 | 17 | 19 | 22 | 14 | 15 | 18 | 14 | 21 | 3 | 9 | 11 |

- Notice:
  - The left child of index i is at index 2*i+1
  - The right child of index i is at index 2*i+2
  - Example: the children of node 3 (19) are 7 (18) and 8 (14)

# Removing and replacing the root

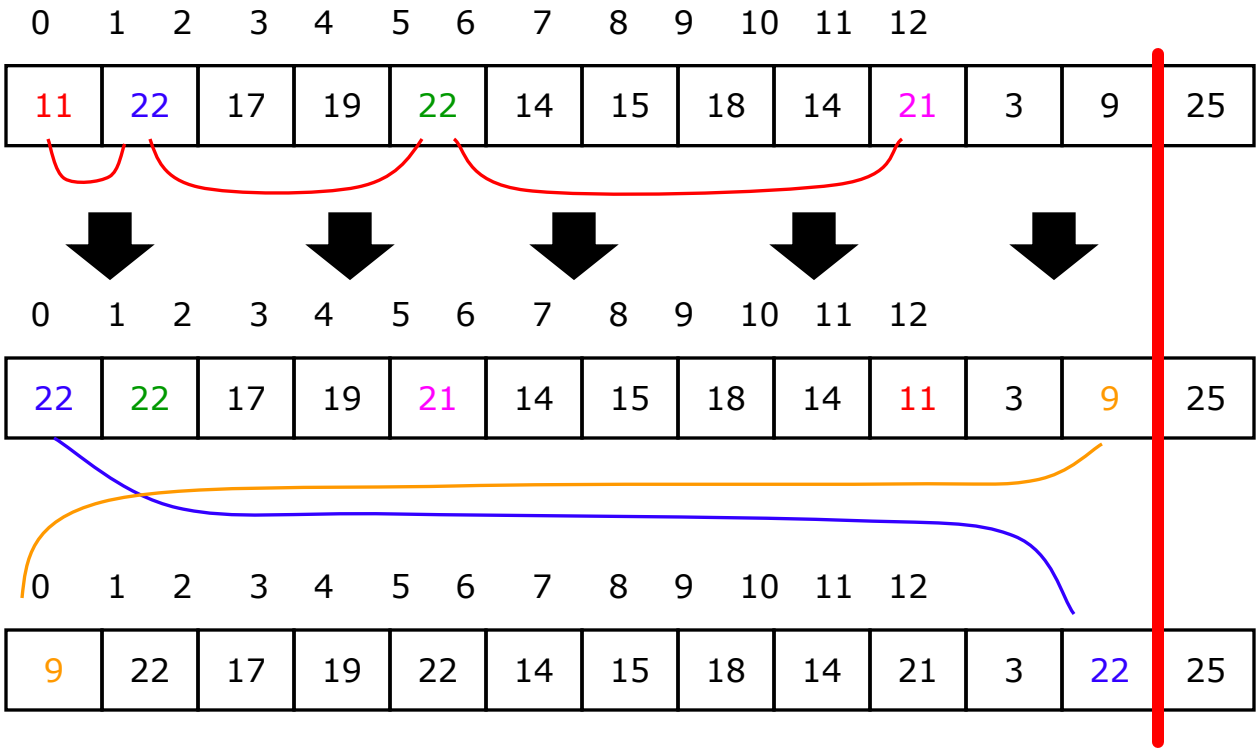
- The “root” is the first element in the array
- The “rightmost node at the deepest level” is the last element
- Swap them...




- ...And pretend that the last element in the array no longer exists—that is, the “last index” is `11` (containing the value 9)

# Reheap and repeat

- Reheap the root node (index 0, containing 11)...




- ...And again, remove and replace the root node
- Remember, though, that the "last" array index is changed
- Repeat until the last becomes first, and the array is sorted!

# Implementation

```
// Ferhat Sal's implementation of Heap Sort. Thanks Ferhat!
package Heap;
import java.util.*;
public class Heap {

    private int[] heaparr = null;

    public Heap() {
        this.heaparr = new int[] { 25, 22, 17, 19, 22, 14, 15, 18, 14, 21, 3, 9, 11};
     // for testing with different data
    //this.heaparr = new int[] { 55,49,38,44,41,32,29,18,11,25,27,5,12,20,13,1};
    //this.heapSort(heaparr);
    }

    private void swap(int[] arr, int index1, int index2) {
        int tmp = arr[index2];
        arr[index2] = arr[index1];
        arr[index1] = tmp;
        System.out.println("swap : " + Arrays.toString(arr));
    }
}
```

```
    public void heapSort(int[] heap) {
        int n = heap.length;
        for (int i = 0; i < n; i++) {
            int wall = n-i-1 ;
            System.out.println("i: " + i +" " +  Arrays.toString(heap));
            swap(heap, 0, n - i - 1);
            int current = 0;
            while (current < wall ) { // loop until we did not reach to "virtual wall" and we have
still children
                int biggest = current ; // set current as biggest
                int left = current * 2 + 1;
                int right = current * 2 + 2;
                // If left child is bigger than root
                if (left < wall && heap[left] > heap[biggest]) {
                    biggest = left;
                }

                // If right child is bigger than biggest
                if (right < wall && heap[right] > heap[biggest]) {
                    biggest = right;
                }
                // If biggest is not current
                if (biggest != current) {
                    swap(heap, current, biggest);
                }
                else {
                    break; // max is current , no need to continue
                }
                current = biggest ;
            }
        }
    }
```

# Analysis I

- Here's how the algorithm starts:

  heapify the array;

- Heapifying the array: we add each of n nodes
  - Each node has to be sifted up, possibly as far as the root
    - Since the binary tree is perfectly balanced, sifting up a single node takes O(log n) time
  - Since we do this n times, heapifying takes n*O(log n) time, that is, O(n log n) time

# Analysis II

- Here's the rest of the algorithm:

```
while the array isn't empty {
    remove and replace the root;
    reheap the new root node;
}
```

- We do the while loop `n` times (actually, `n-1` times), because we remove one of the `n` nodes each time
- Removing and replacing the root takes `O(1)` time
- Therefore, the total time is `n` times however long it takes the `reheap` method

# Analysis III

- To reheap the root node, we have to follow *one path* from the root to a leaf node (and we might stop before we reach a leaf)
- The binary tree is perfectly balanced
- Therefore, this path is O(log n) long
  - And we only do O(1) operations at each node
  - Therefore, reheaping takes O(log n) times
- Since we reheap inside a while loop that we do n times, the total time for the while loop is n*O(log n), or O(n log n)

# Analysis IV

- Here's the algorithm again:

```
heapify the array;
while the array isn't empty {
    remove and replace the root;
    reheap the new root node;
}
```

- We have seen that heapifying takes O(n log n) time
- The while loop takes O(n log n) time
- The total time is therefore O(n log n) + O(n log n)
- This is the same as O(n log n) time

# The End