Graph Algorithms

adapted from David Luebke

Graph Algorithms

- Graph algorithms
  - Should be largely review, easier for exam

Graphs

- A graph G = (V, E)
  - V = set of vertices
  - E = set of edges = subset of $V \times V$
  - Thus $|E| = O(|V|^2)$

Graph Variations

- Variations:
  - A *connected graph* has a path from every vertex to every other
  - In an *undirected graph:*
    - Edge (u,v) = edge (v,u)
    - No self-loops
  - In a *directed* graph:
    - Edge (u,v) goes from vertex u to vertex v, notated u→v

Graph Variations

- More variations:
  - A *weighted graph* associates weights with either the edges or the vertices
    - E.g., a road map: edges might be weighted w/ distance
  - A *multigraph* allows multiple edges between the same vertices
    - E.g., the call graph in a program (a function can get called from multiple points in another function)

Graphs

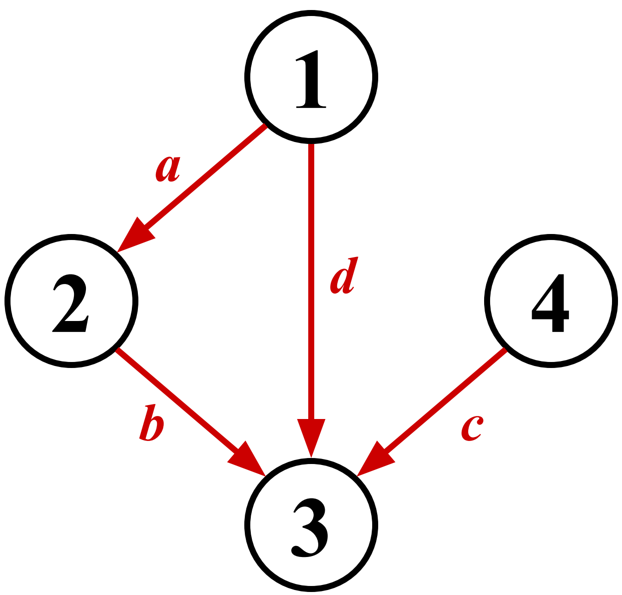
- We will typically express running times in terms of |E| and |V| (often dropping the |'s)
  - If $|E| \approx |V|^2$ the graph is *dense*
  - If $|E| \approx |V|$ the graph is *sparse*
- If you know you are dealing with dense or sparse graphs, different data structures may make sense

Representing Graphs

- Assume V = $\{1, 2, \ldots, n\}$
- An *adjacency matrix* represents the graph as a n x n matrix A:
  - A[i, j] = 1 if edge $(i, j) \in$ E (or weight of edge)
    = 0 if edge $(i, j) \notin$ E

Graphs: Adjacency Matrix

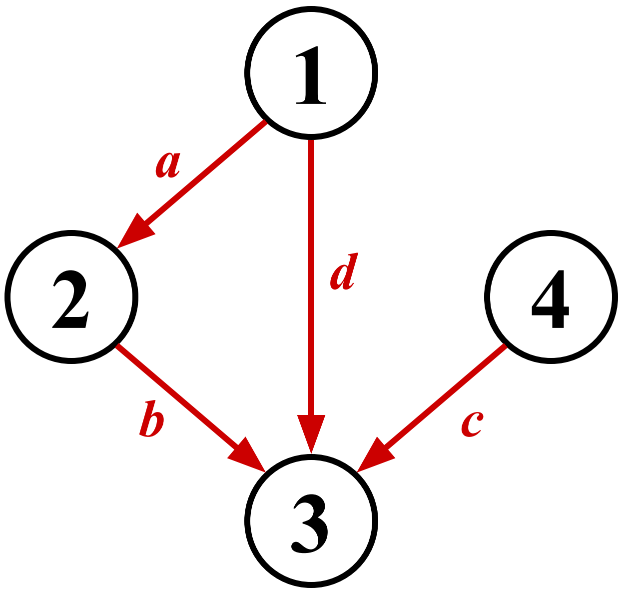
- Example:



A	1	2	3	4
1				
2				
3			??	
4				

Graphs: Adjacency Matrix

- Example:



A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

Graphs: Adjacency Matrix

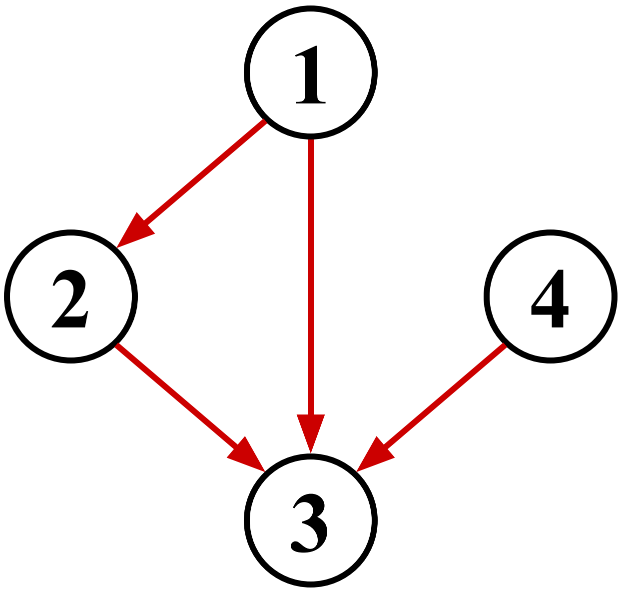
- *How much storage does the adjacency matrix require?*
  - A: $O(V^2)$
- *What is the minimum amount of storage needed by an adjacency matrix representation of an undirected graph with 4 vertices?*
  - A: 6 bits
    - Undirected graph → matrix is symmetric
    - No self-loops → don't need diagonal

Graphs: Adjacency Matrix

- The adjacency matrix is a dense representation
  - Usually too much storage for large graphs
  - But can be very efficient for small graphs
- Most large interesting graphs are sparse
  - E.g., planar graphs, in which no edges cross, have $|E| = O(|V|)$ by Euler's formula
  - For this reason the *adjacency list* is often a more appropriate representation

Graphs: Adjacency List

- Adjacency list: for each vertex $v \in V$, store a list of vertices adjacent to v
- Example:
  - Adj[1] = {2,3}
  - Adj[2] = {3}
  - Adj[3] = {}
  - Adj[4] = {3}



- Variation: can also keep a list of edges coming *into* vertex

Graphs: Adjacency List

- How much storage is required?
  - The *degree* of a vertex v = # incident edges
    - Directed graphs have in-degree, out-degree
  - For directed graphs, # of items in adjacency lists is

    Σ out-degree(v) = |E|

    takes $\Theta(V + E)$ storage (*Why?*)
  - For undirected graphs, # items in adj lists is

    Σ degree(v) = 2 |E| (*handshaking lemma*)

    also $\Theta(V + E)$ storage
- So: Adjacency lists take O(V+E) storage

Graph Searching

- Given: a graph $G = (V, E)$, directed or undirected
- Goal: methodically explore every vertex and every edge
- Ultimately: build a tree on the graph
  - Pick a vertex as the root
  - Choose certain edges to produce a tree
  - Note: might also build a *forest* if graph is not connected

Breadth-First Search

- "Explore" a graph, turning it into a tree
  - One vertex at a time
  - Expand frontier of explored vertices across the *breadth* of the frontier
- Builds a tree over the graph
  - Pick a *source vertex* to be the root
  - Find ("discover") its children, then their children, etc.

Breadth-First Search

- Again will associate vertex “colors” to guide the algorithm
  - White vertices have not been discovered
    - All vertices start out white
  - Grey vertices are discovered but not fully explored
    - They may be adjacent to white vertices
  - Black vertices are discovered and fully explored
    - They are adjacent only to black and gray vertices
- Explore vertices by scanning adjacency list of grey vertices

Breadth-First Search

```
BFS(G, s) {
    initialize vertices;
    Q = {s};     // Q is a queue (duh); initialize to s
    while (Q not empty) {
        u = RemoveTop(Q);
        for each v ∈ u->adj {
            if (v->color == WHITE)
                v->color = GREY;
                v->d = u->d + 1;
                v->p = u;
                Enqueue(Q, v);
        }
        u->color = BLACK;
    }
}
```

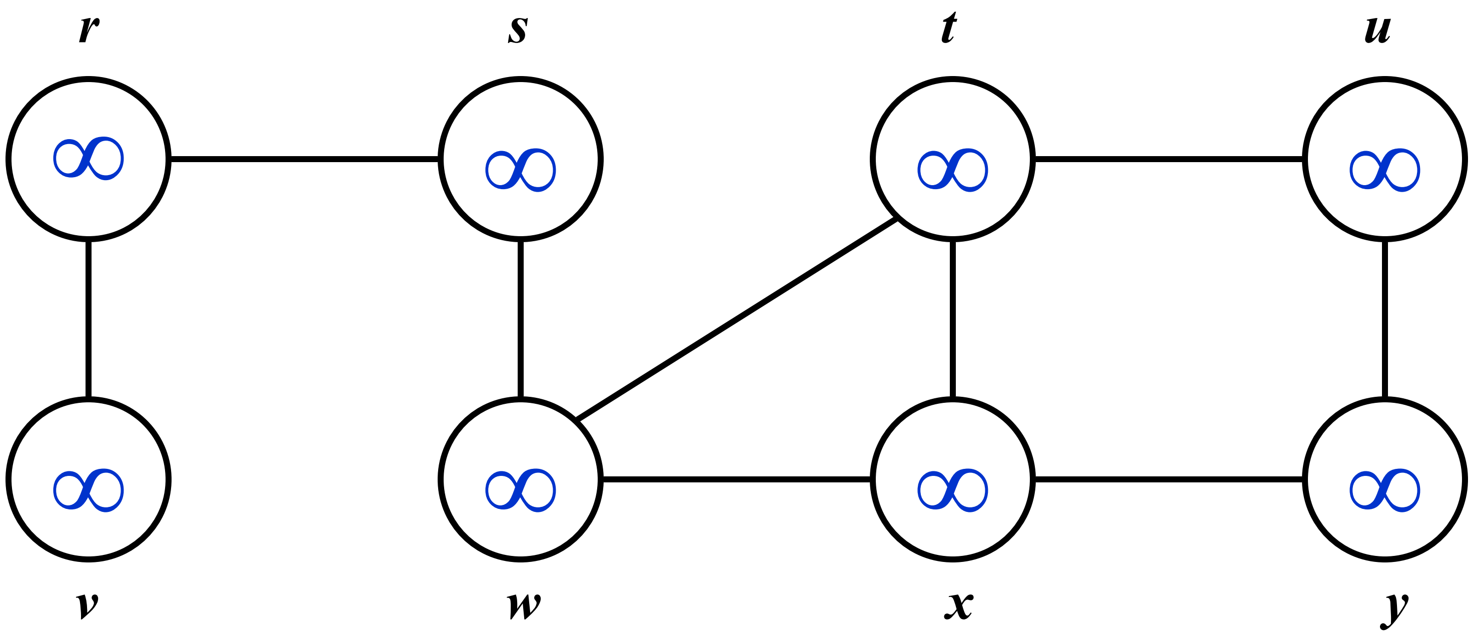
What does **`v->d`** *represent?*

What does **`v->p`** *represent?*

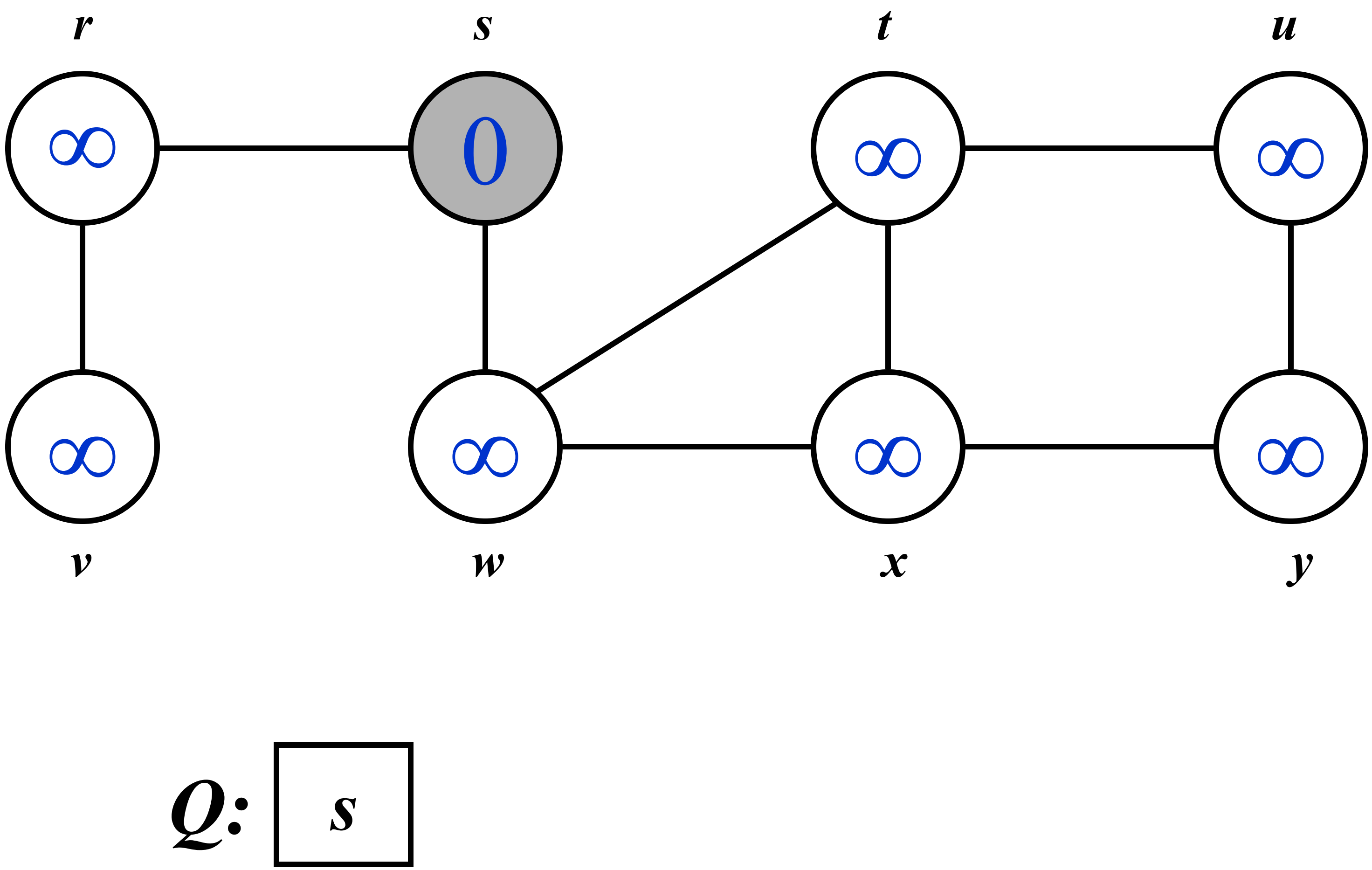
Breadth-First Search

```
BFS(G, s)
for each vertex u ∈ V[G] − {s}
      do color[u] ← WHITE
         d[u] ← ∞
         π[u] ← NIL
color[s] ← GRAY
d[s] ← 0
π[s] ← NIL
Q ← Ø
ENQUEUE(Q, s)
while Q ≠ Ø
      do u ← DEQUEUE(Q)
         for each v ∈ Adj[u]
             do if color[v] = WHITE
                   then color[v] ← GRAY
                        d[v] ← d[u] + 1
                        π[v] ← u
                        ENQUEUE(Q, v)
         color[u] ← BLACK
```

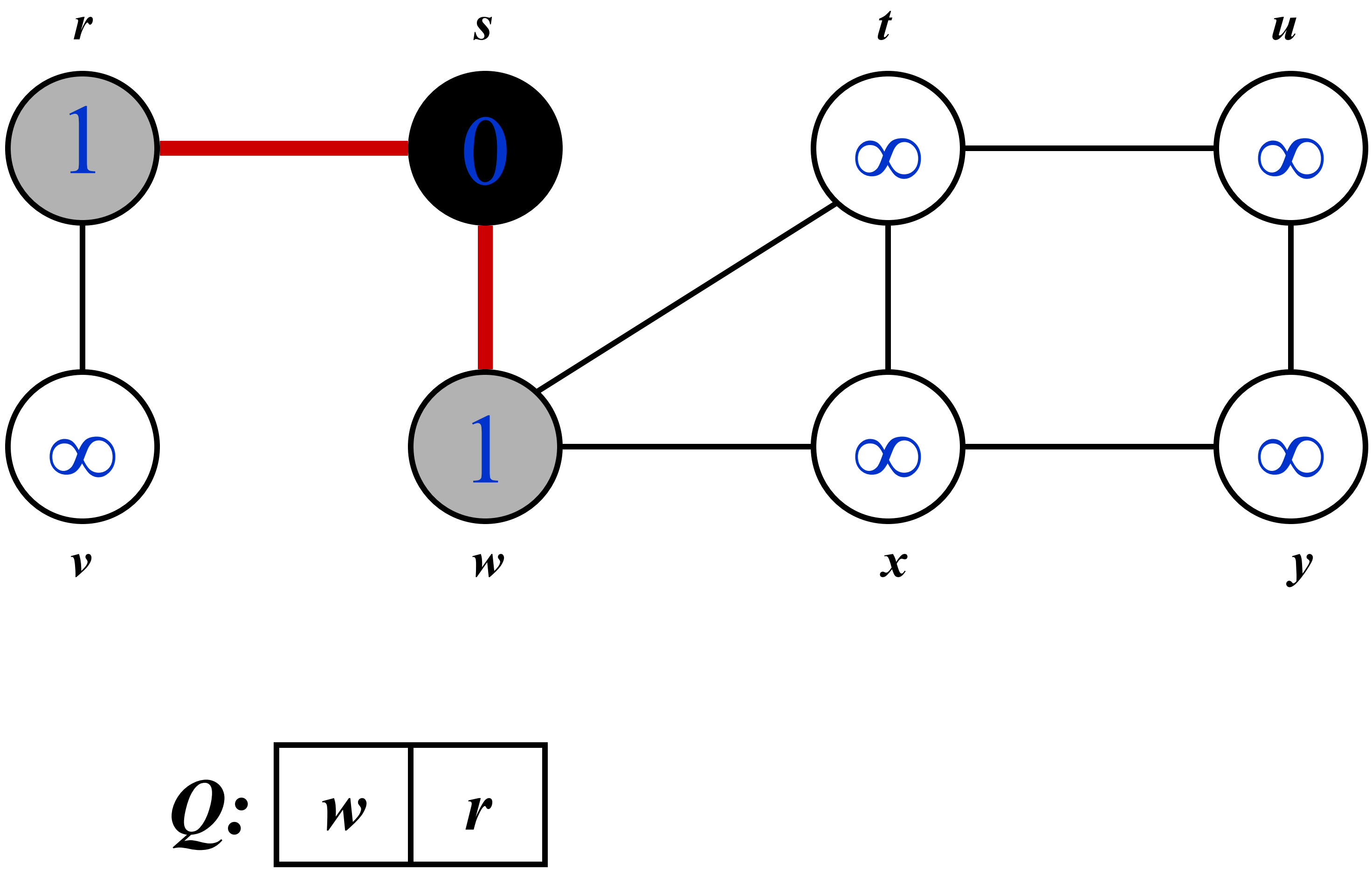
Breadth-First Search: Example



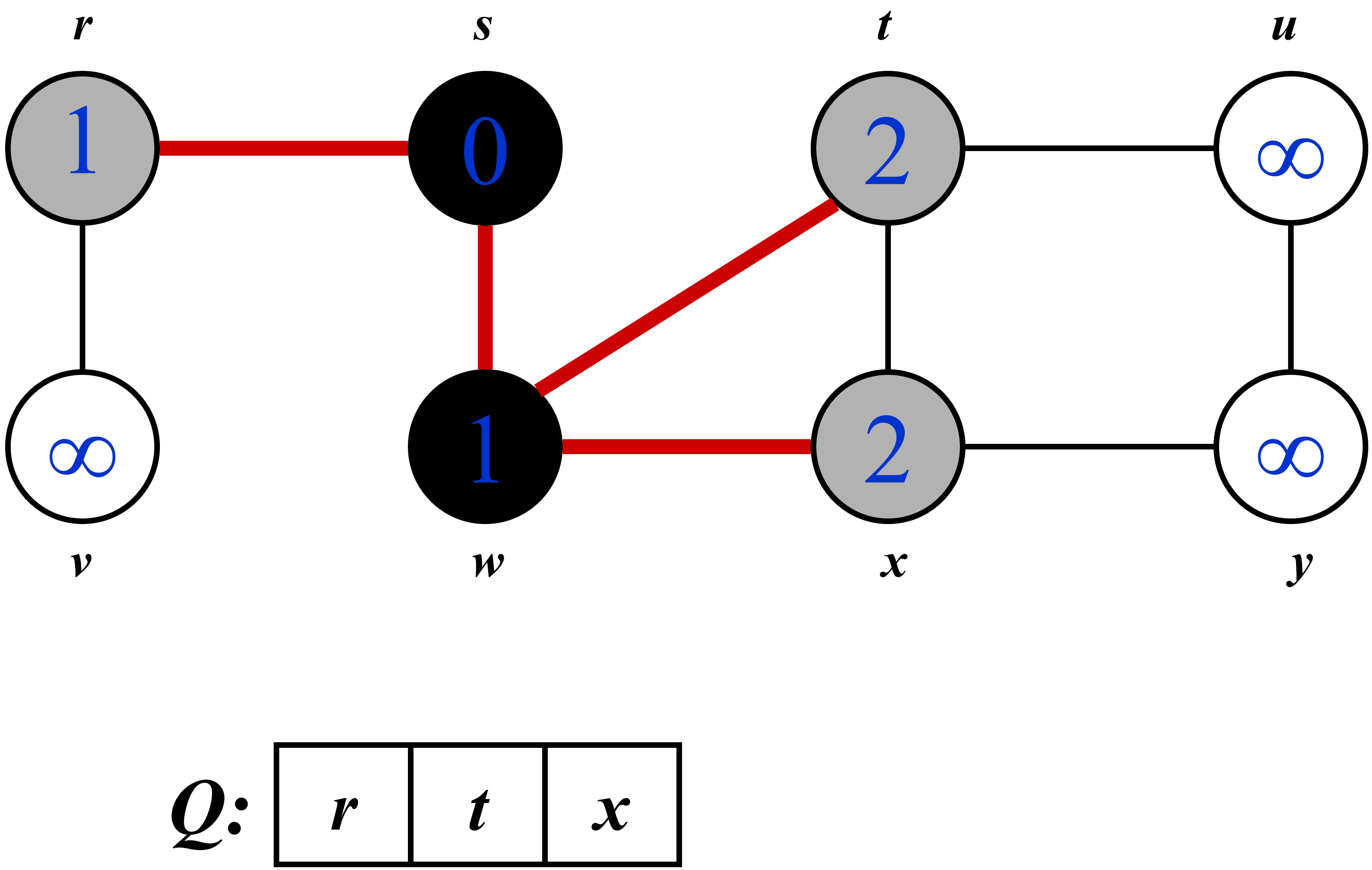
Breadth-First Search: Example



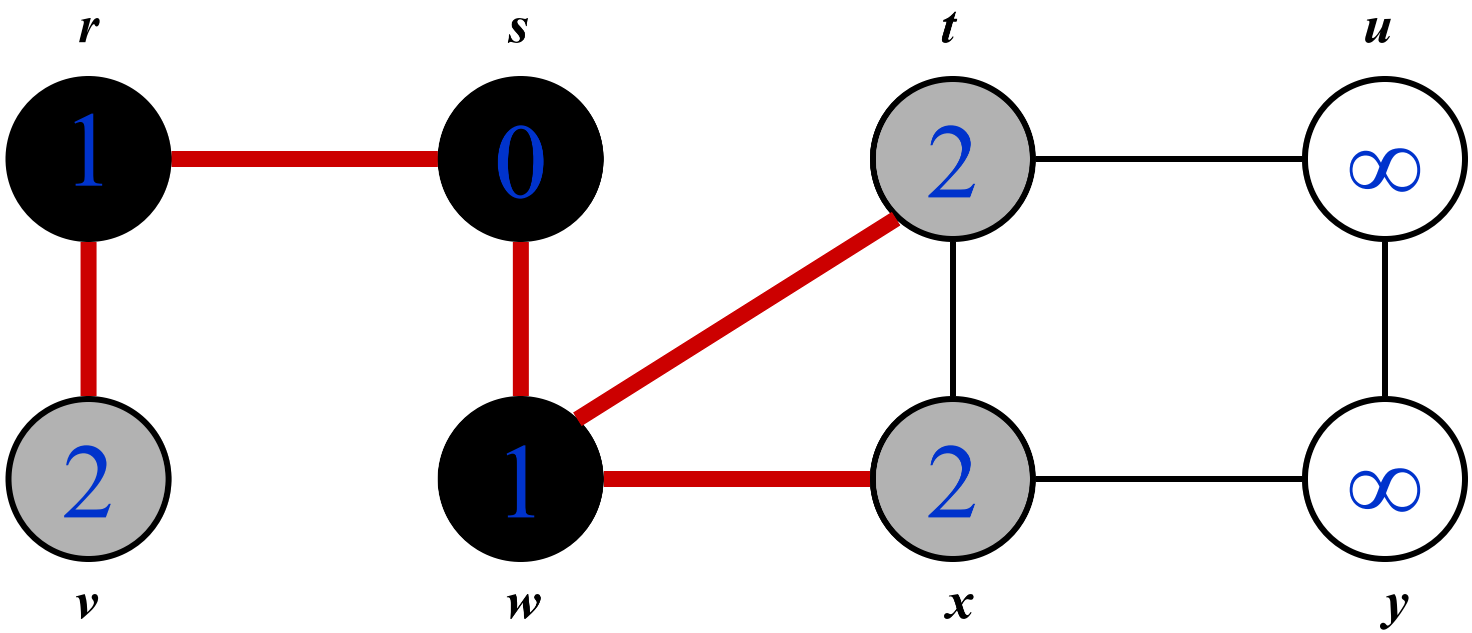
Breadth-First Search: Example



Breadth-First Search: Example

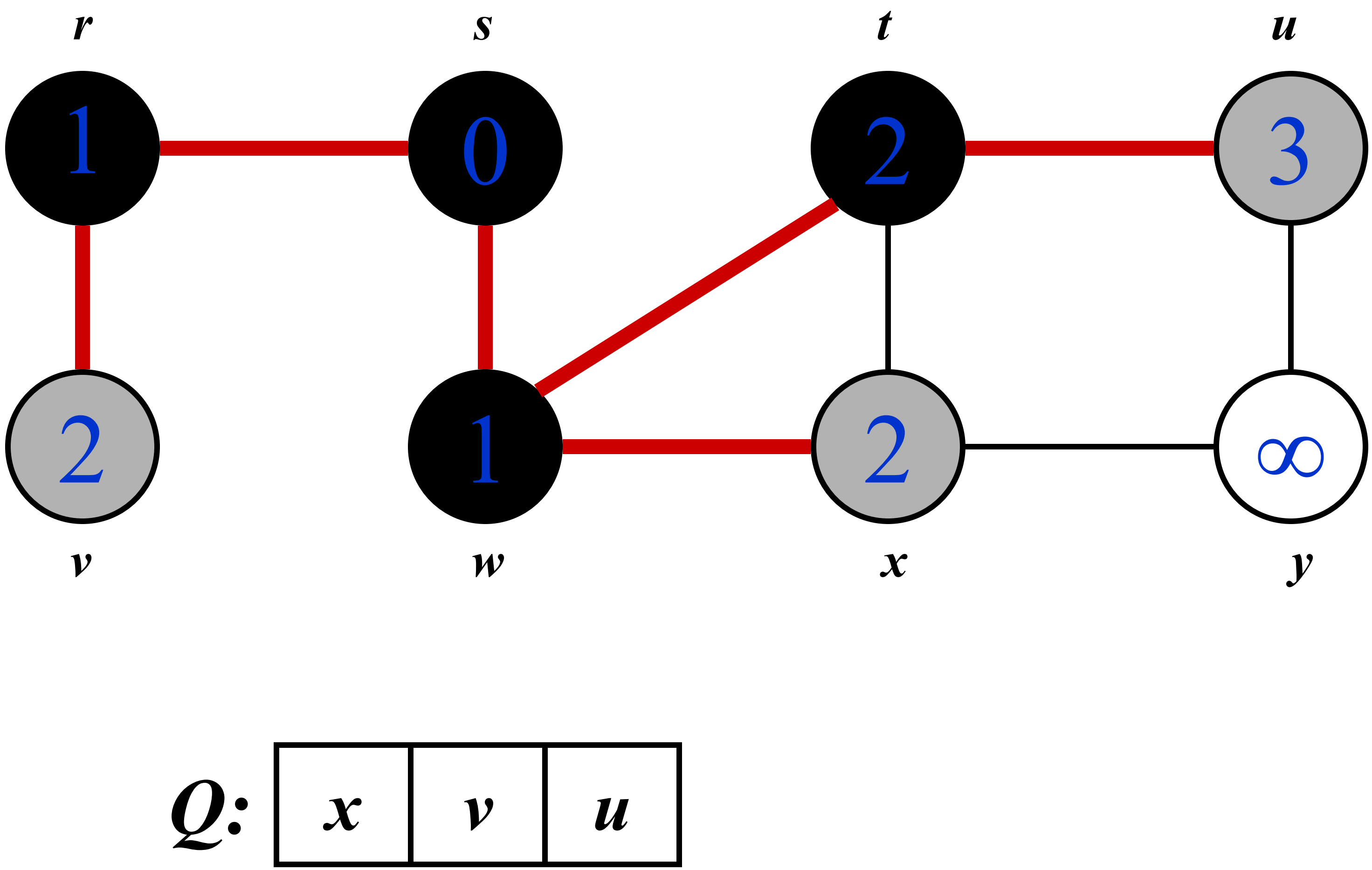


Breadth-First Search: Example

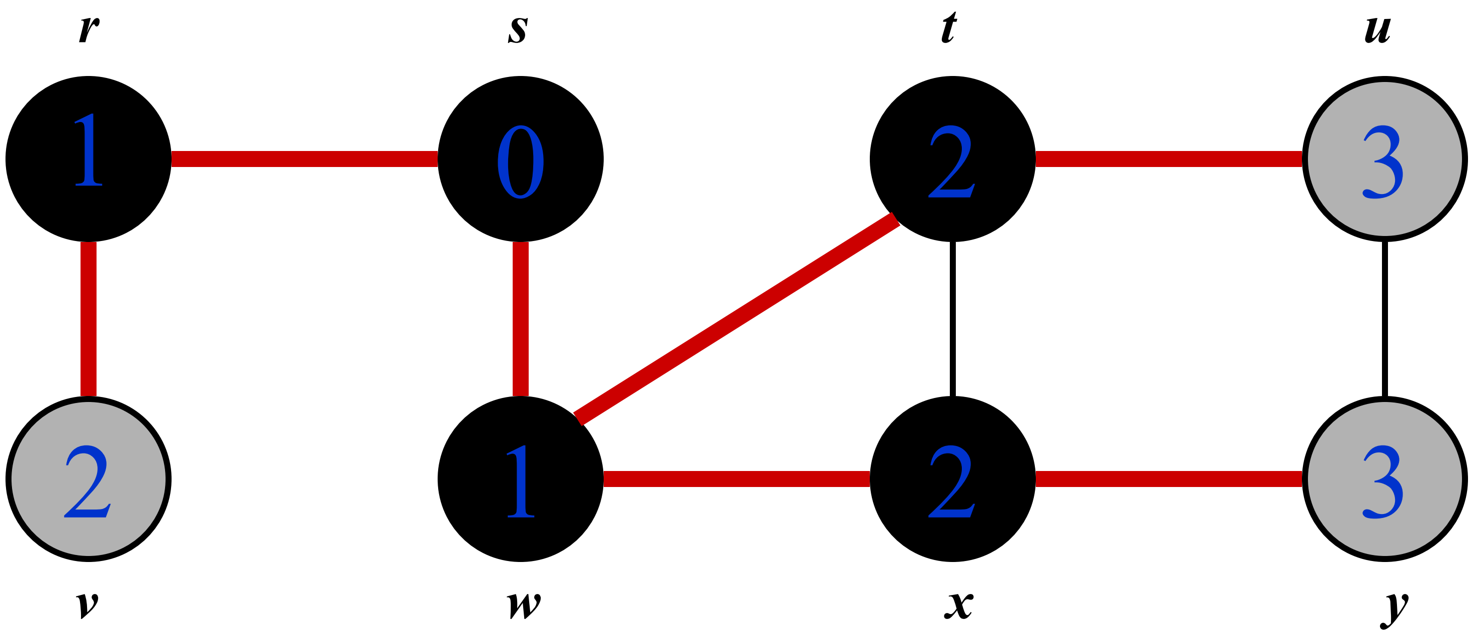


Q: | *t* | *x* | *v* |

Breadth-First Search: Example

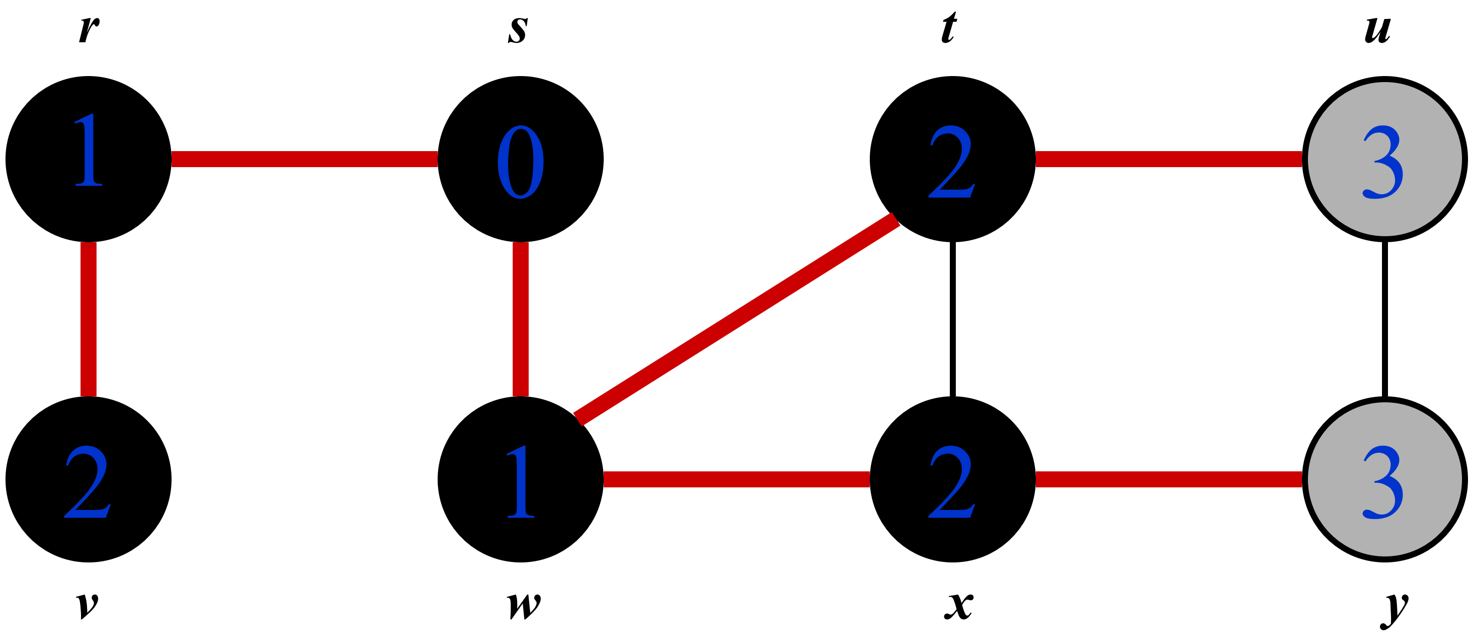


Breadth-First Search: Example



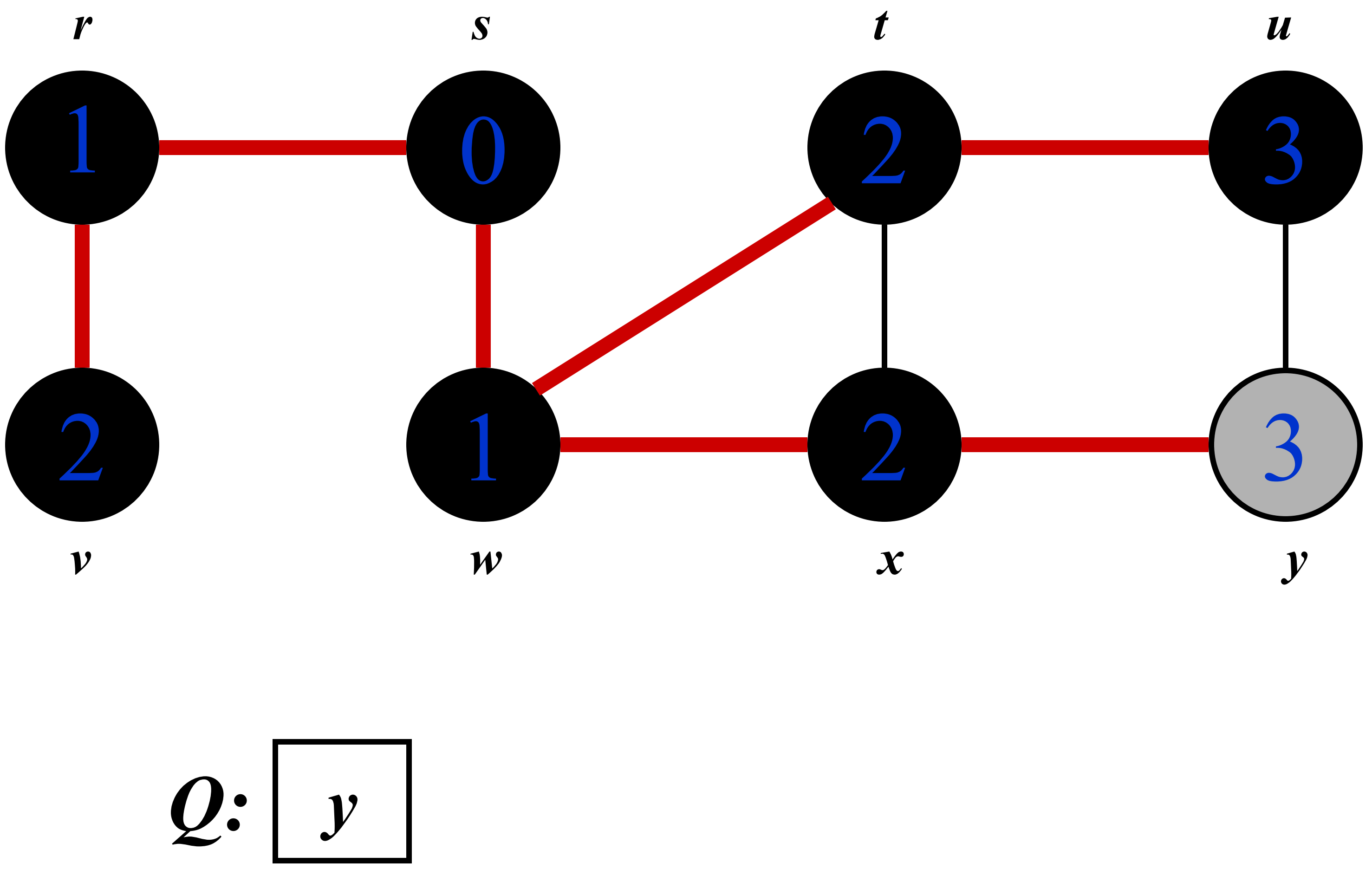
Q: | *v* | *u* | *y* |

Breadth-First Search: Example

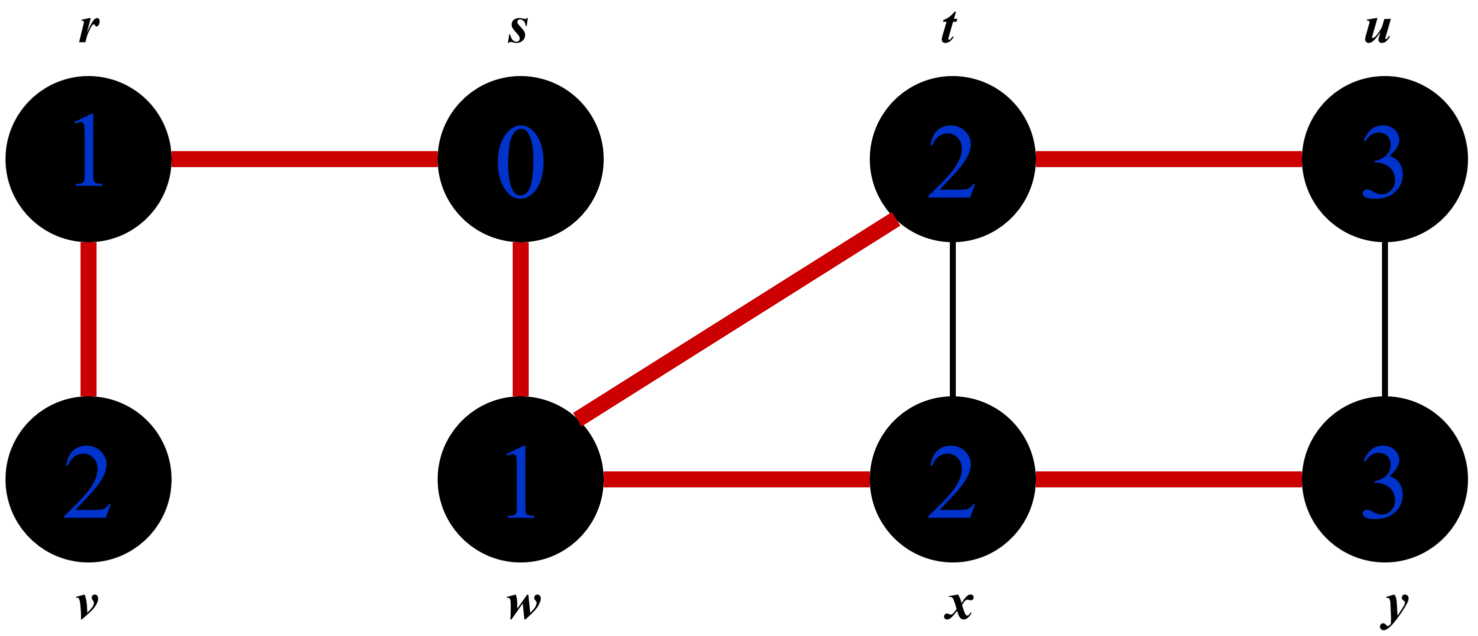


Q:	*u*	*y*

Breadth-First Search: Example



Breadth-First Search: Example



Q: Ø

BFS: The Code Again

```
BFS(G, s) {
    initialize vertices;
    Q = {s};
    while (Q not empty) {
        u = RemoveTop(Q);
        for each v ∈ u->adj {
            if (v->color == WHITE)
                v->color = GREY;
                v->d = u->d + 1;
                v->p = u;
                Enqueue(Q, v);
        }
        u->color = BLACK;
    }
}
```

Touch every vertex: O(V)

u = every vertex, but only once

(Why?)

So v = every vertex that appears in some other vert's adjacency list

What will be the running tim

Total running time: O(V+E)

BFS: The Code Again

```
BFS(G, s) {
    initialize vertices;
    Q = {s};
    while (Q not empty) {
        u = RemoveTop(Q);
        for each v ∈ u->adj {
            if (v->color == WHITE)
                v->color = GREY;
                v->d = u->d + 1;
                v->p = u;
                Enqueue(Q, v);
        }
        u->color = BLACK;
    }
}
```

What will be the storage cost in addition to storing the tree?

Total space used:
O(max(degree(v))) = O(E)

Breadth-First Search: Properties

- BFS calculates the *shortest-path distance* to the source node
  - Shortest-path distance $\delta(s,v)$ = minimum number of edges from s to v, or ∞ if v not reachable from s
  - Proof given in the book (p. 472-5)
- BFS builds *breadth-first tree*, in which paths to root represent shortest paths in G
  - Thus can use BFS to calculate shortest path from one vertex to another in $O(V+E)$ time

Depth-First Search

- *Depth-first search* is another strategy for exploring a graph
  - Explore "deeper" in the graph whenever possible
  - Edges are explored out of the most recently discovered vertex v that still has unexplored edges
  - When all of v's edges have been explored, backtrack to the vertex from which v was discovered

DFS Code

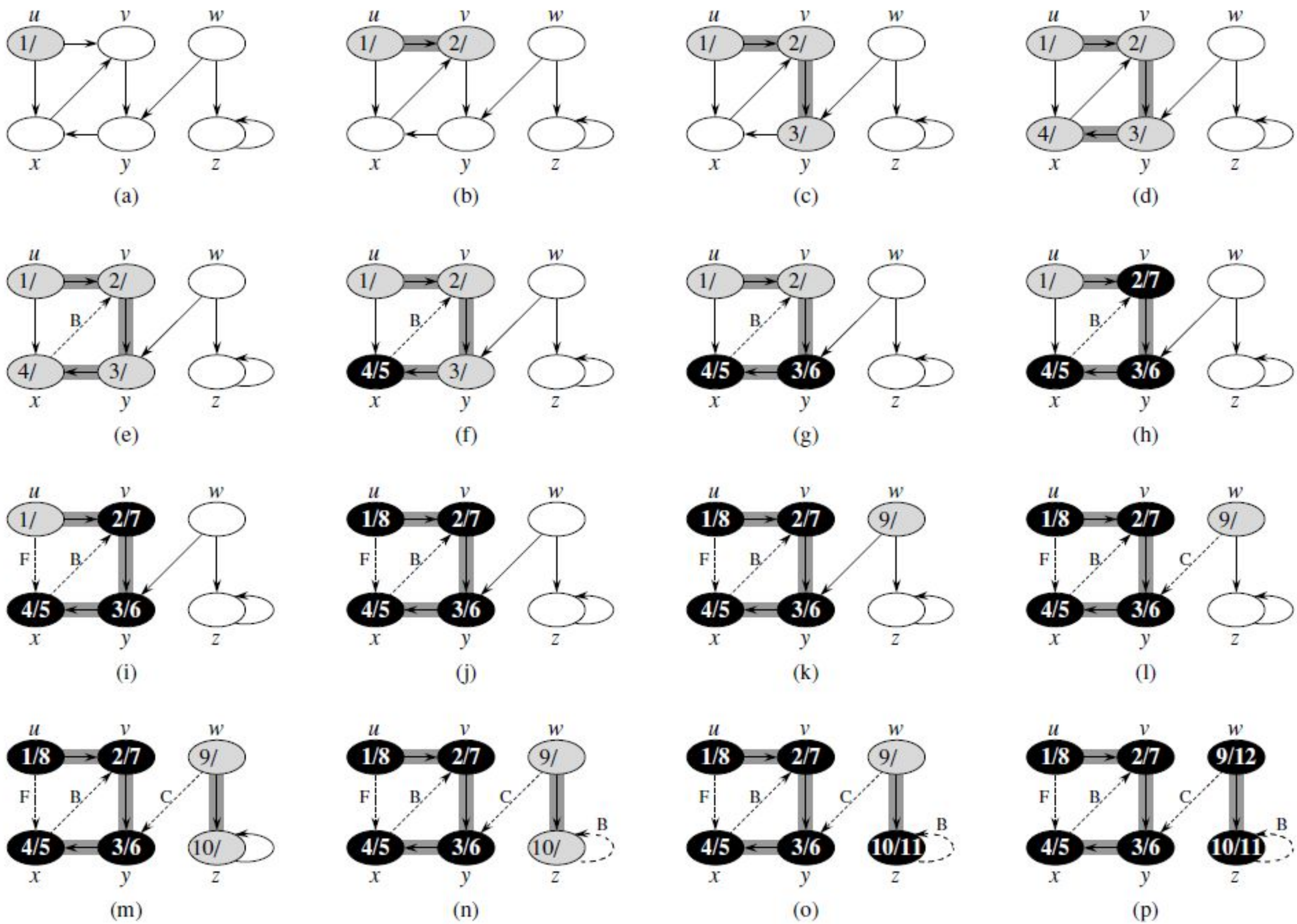
```
DFS(G)
{
    for each vertex u ∈ G->V
    {
        u->color = WHITE;
    }
    time = 0;
    for each vertex u ∈ G->V
    {
        if (u->color == WHITE)
            DFS_Visit(u);
    }
}
```

```
DFS_Visit(u)
{
    u->color = YELLOW;
    time = time+1;
    u->d = time;
    for each v ∈ u->Adj[]
    {
        if (v->color == WHITE)
            DFS_Visit(v);
    }
    u->color = BLACK;
    time = time+1;
    u->f = time;
}
```

DFS Code

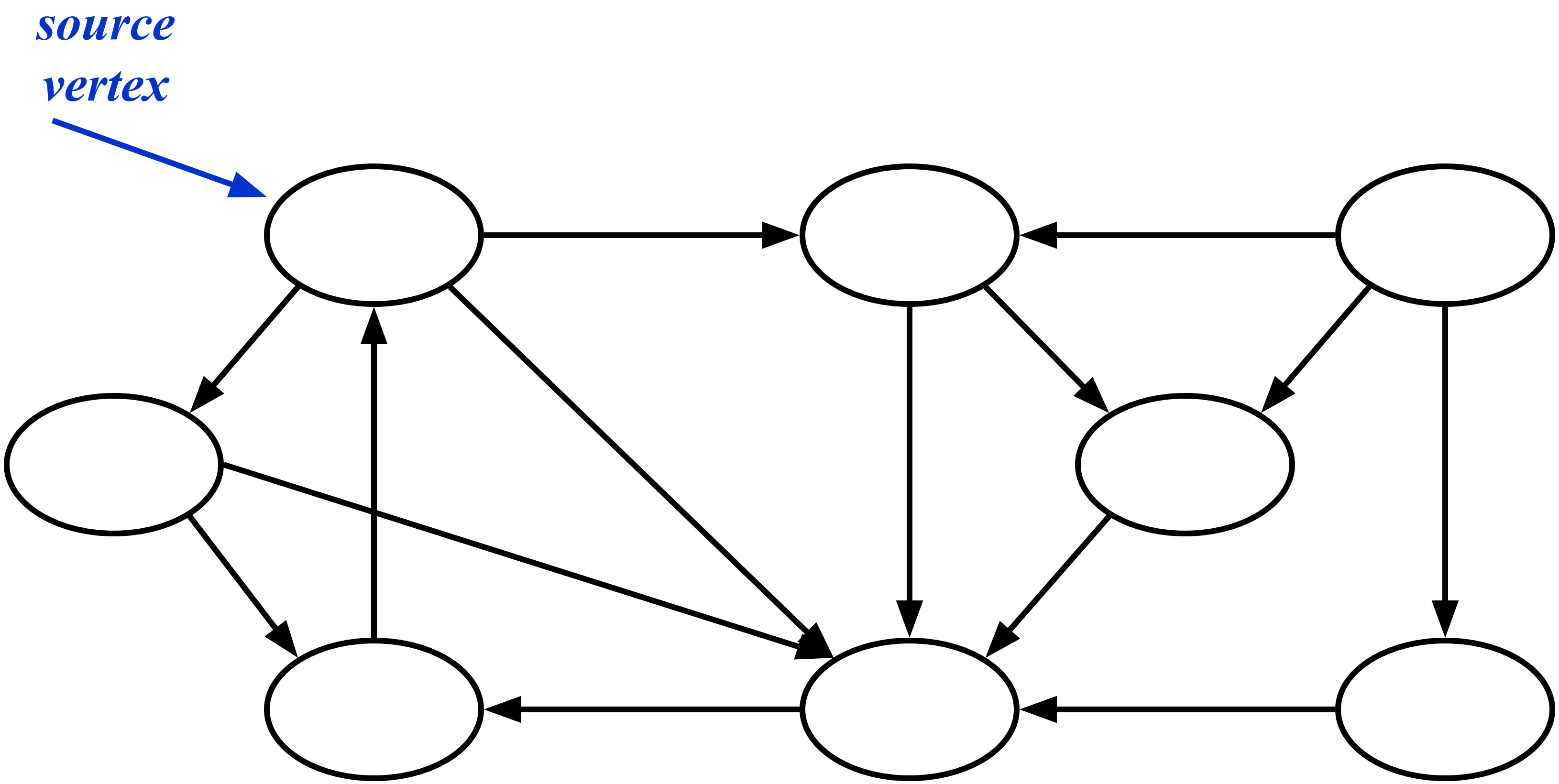
```
DFS(G)
for each vertex u ∈ V[G]
        do color[u] ← WHITE
           π[u] ← NIL
time ← 0
for each vertex u ∈ V[G]
        do if color[u] = WHITE
              then DFS-VISIT(u)
```

```
DFS-VISIT(u)
color[u] ← GRAY              ▷ White vertex u has just been discovered.
time ← time + 1
d[u] ← time
for each v ∈ Adj[u]          ▷ Explore edge (u, v).
        do if color[v] = WHITE
              then π[v] ← u
                   DFS-VISIT(v)
color[u] ← BLACK             ▷ Blacken u; it is finished.
f[u] ← time ← time + 1
```

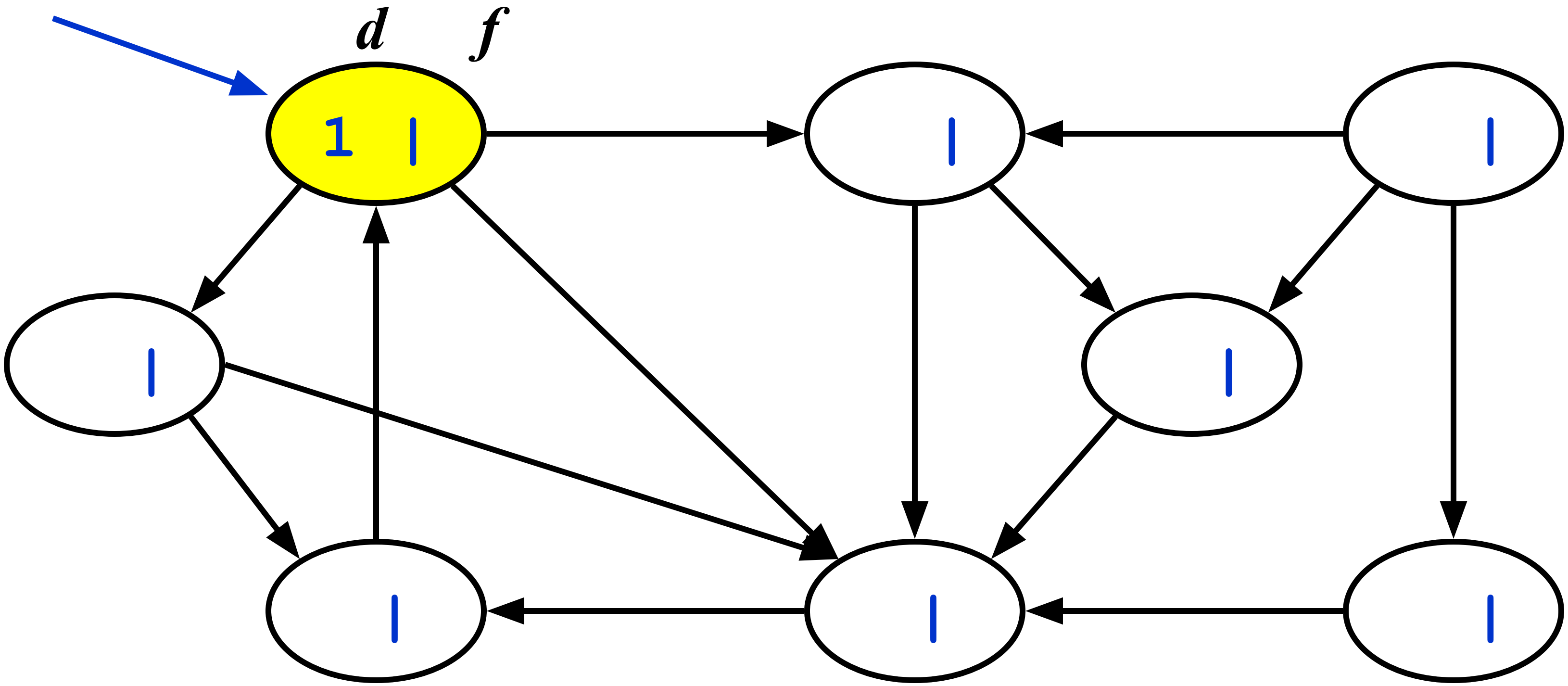


(a) u v w x y z 1/
(b) u v w x y z 1/ 2/
(c) u v w x y z 1/ 2/ 3/
(d) u v w x y z 1/ 2/ 4/ 3/
(e) u v w x y z 1/ 2/ B 4/ 3/
(f) u v w x y z 1/ 2/ B 4/5 3/
(g) u v w x y z 1/ 2/ B 4/5 3/6
(h) u v w x y z 1/ 2/7 B 4/5 3/6
(i) u v w x y z 1/ 2/7 F B 4/5 3/6
(j) u v w x y z 1/8 2/7 F B 4/5 3/6
(k) u v w x y z 1/8 2/7 9/ F B 4/5 3/6
(l) u v w x y z 1/8 2/7 9/ F B C 4/5 3/6
(m) u v w x y z 1/8 2/7 9/ F B C 4/5 3/6 10/
(n) u v w x y z 1/8 2/7 9/ F B C B 4/5 3/6 10/
(o) u v w x y z 1/8 2/7 9/ F B C B 4/5 3/6 10/11
(p) u v w x y z 1/8 2/7 9/12 F B C B 4/5 3/6 10/11

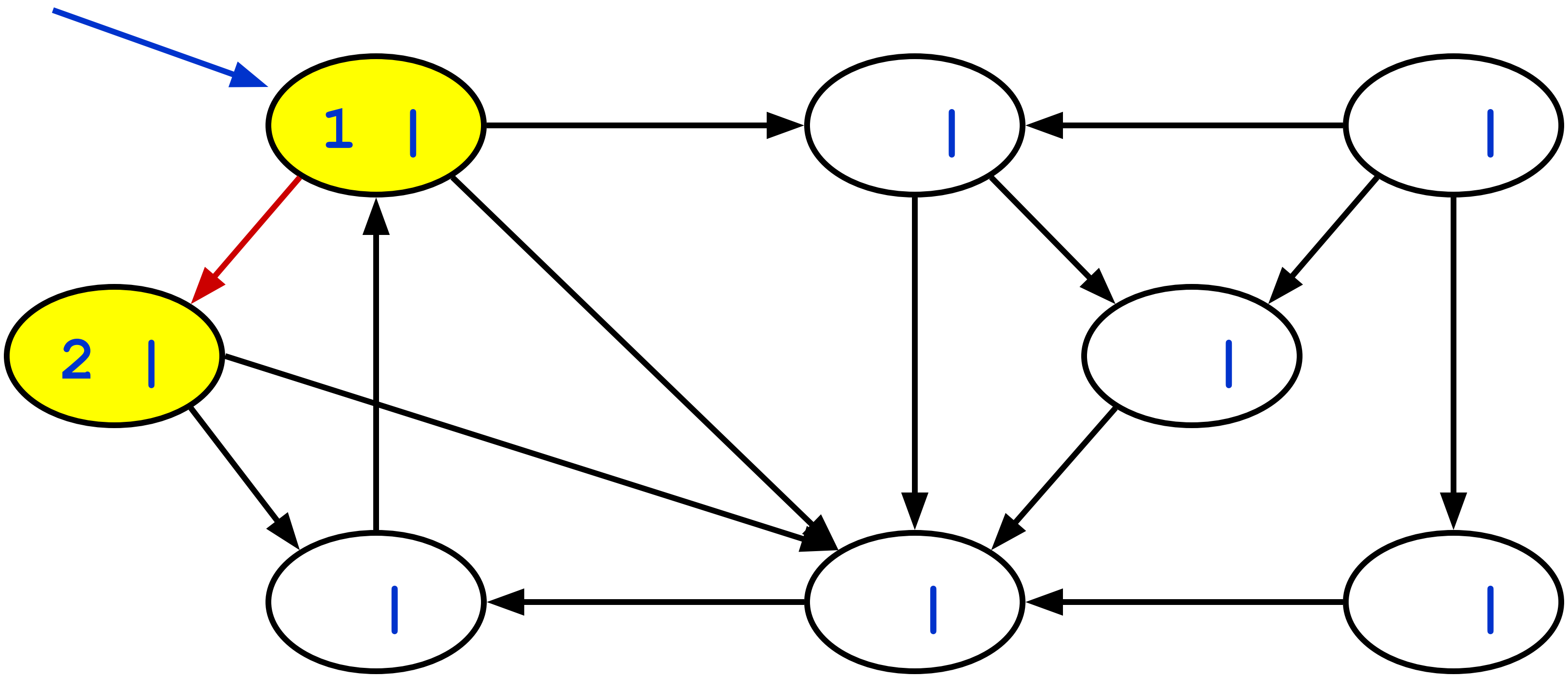
DFS Example



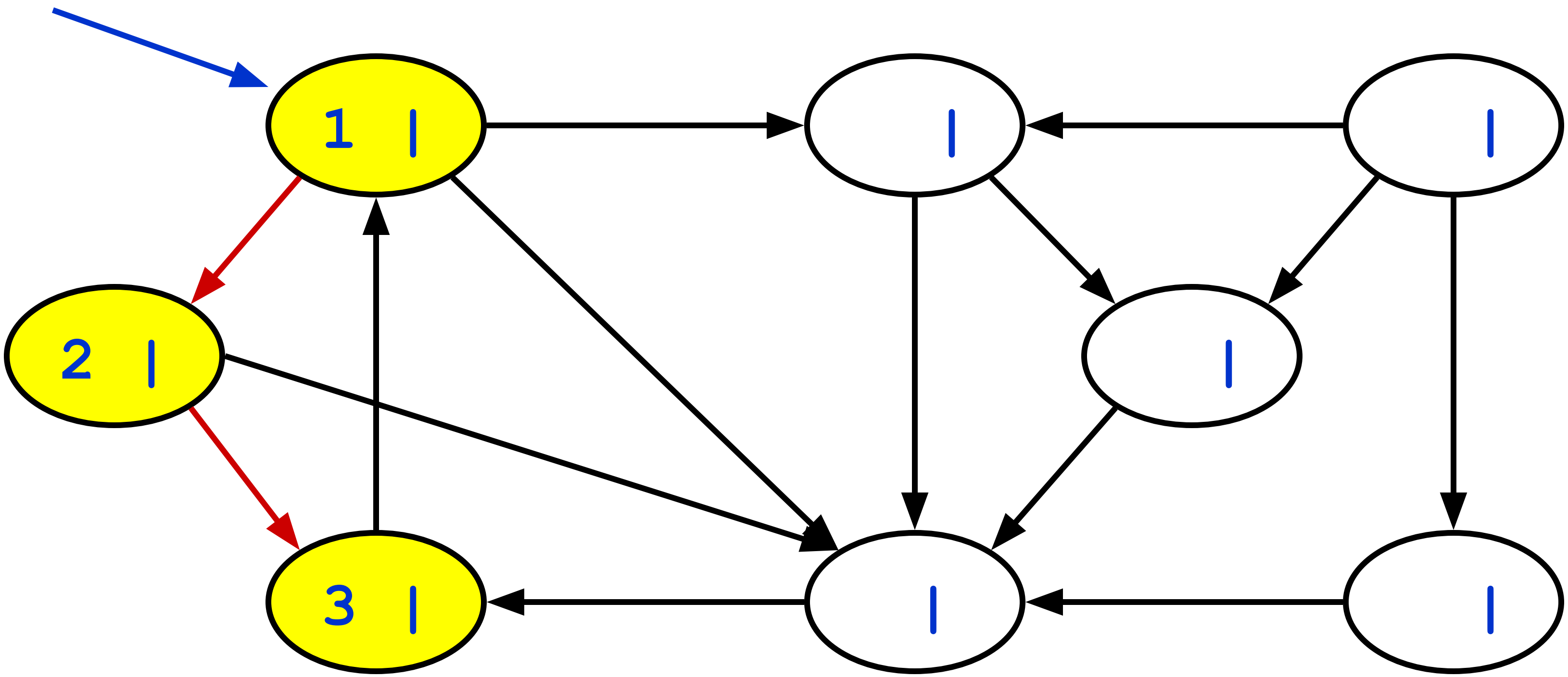
DFS Example



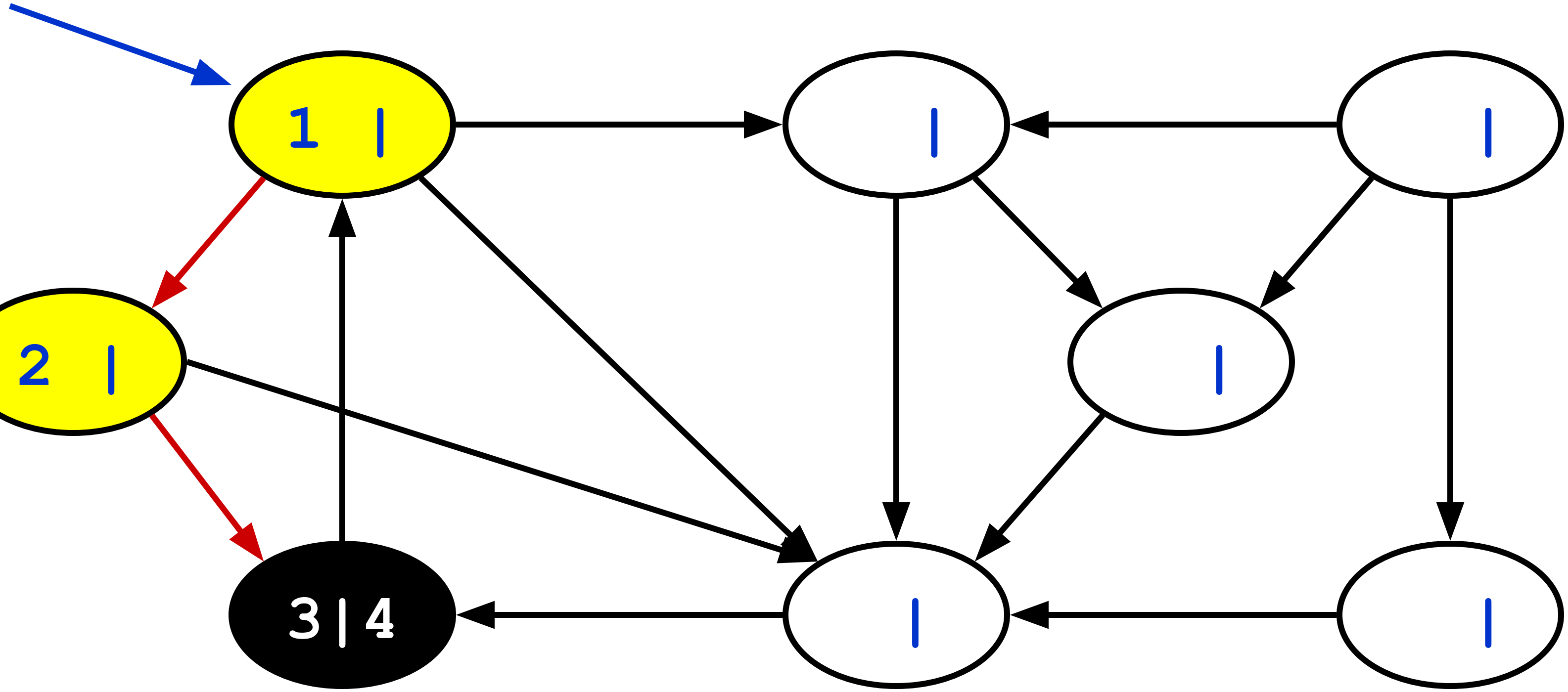
DFS Example



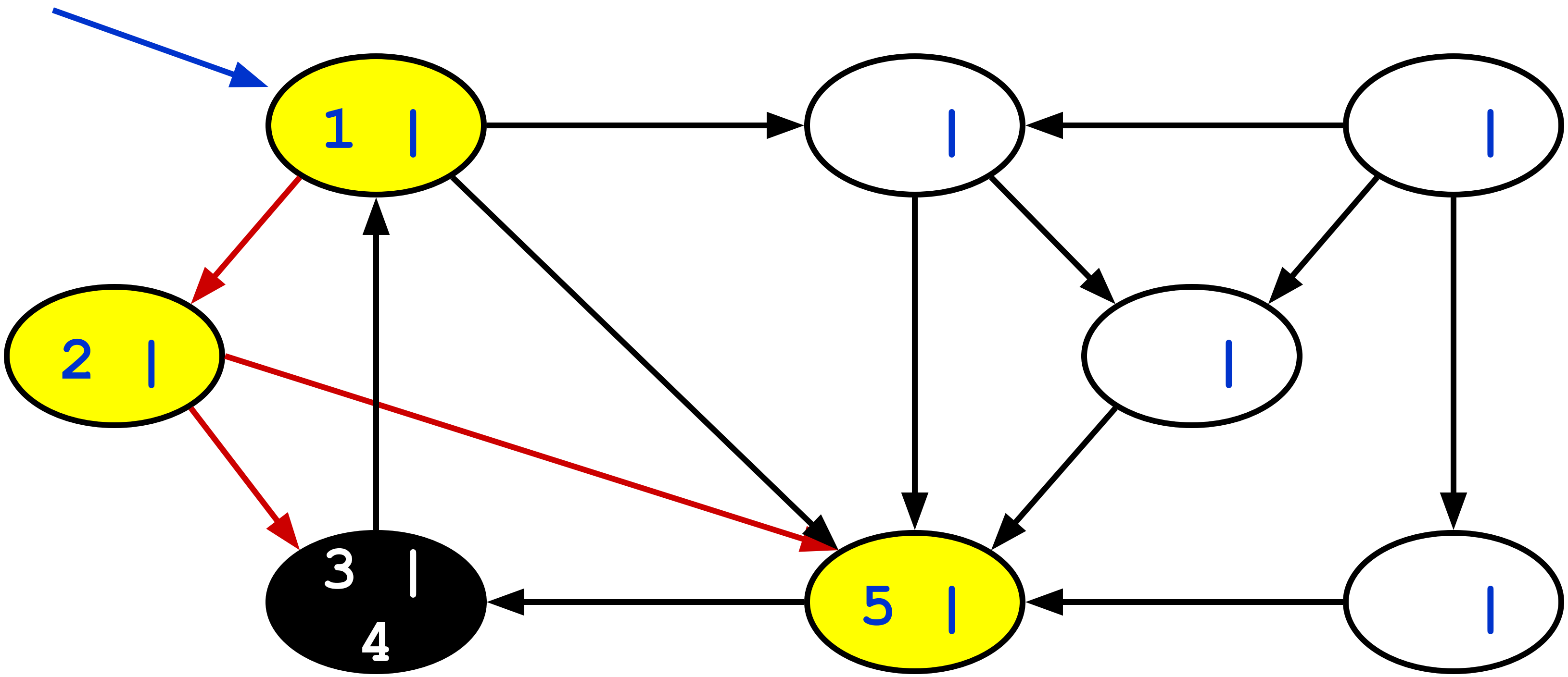
DFS Example



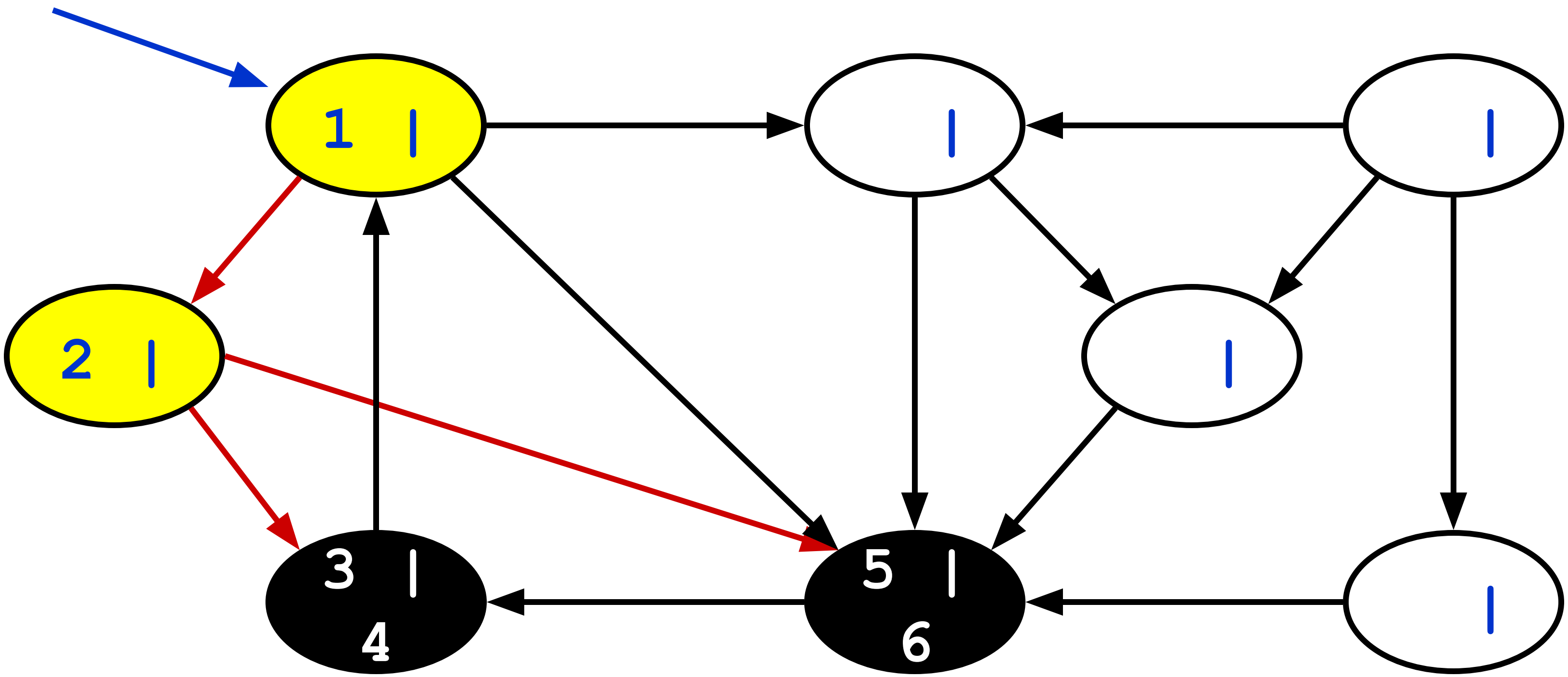
DFS Example



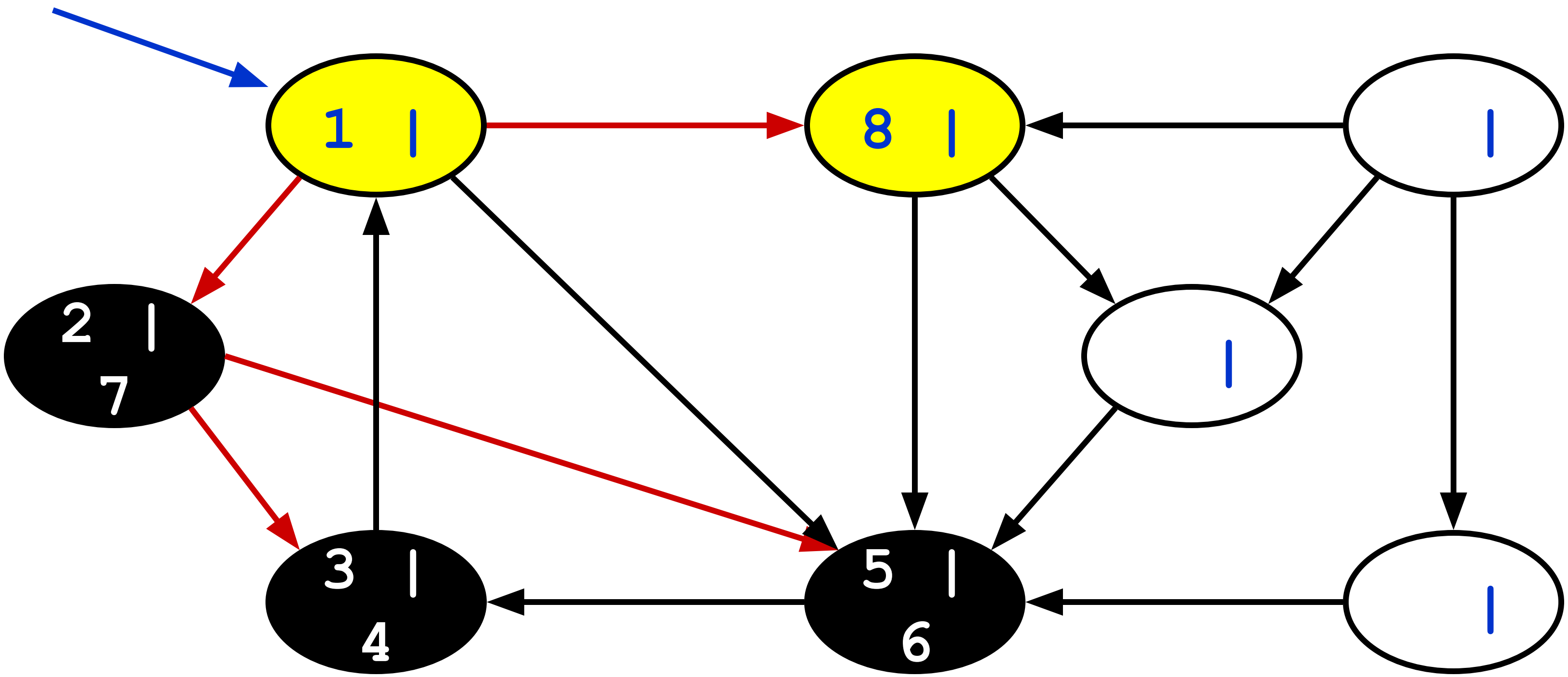
DFS Example



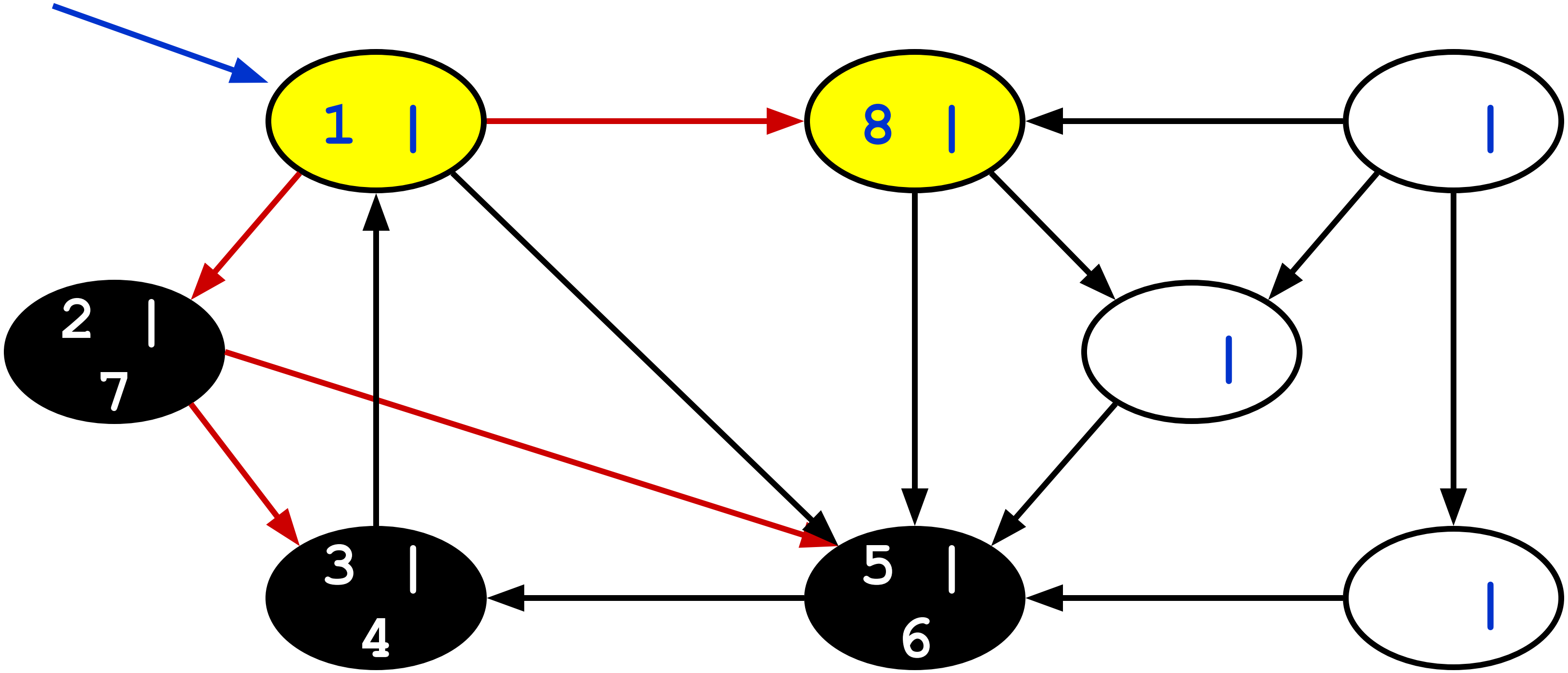
DFS Example



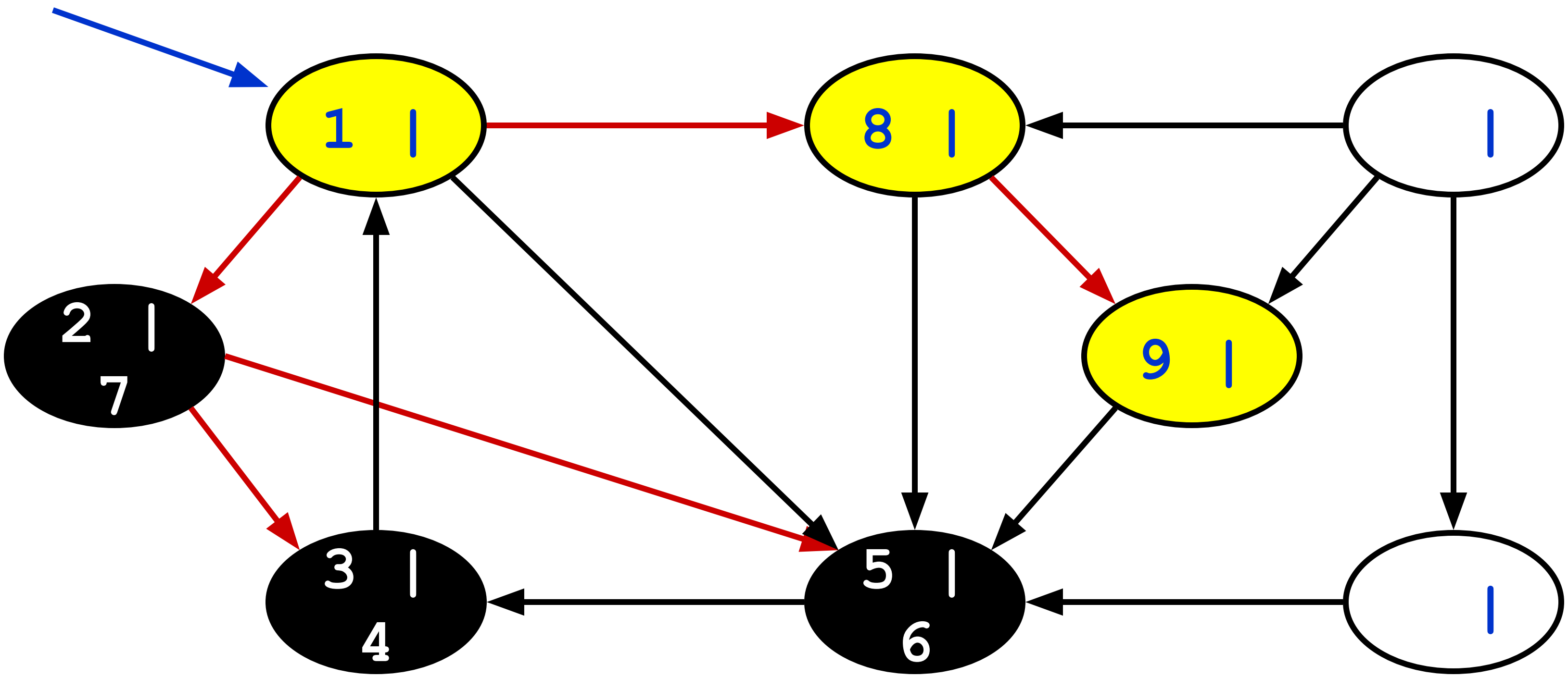
DFS Example



DFS Example

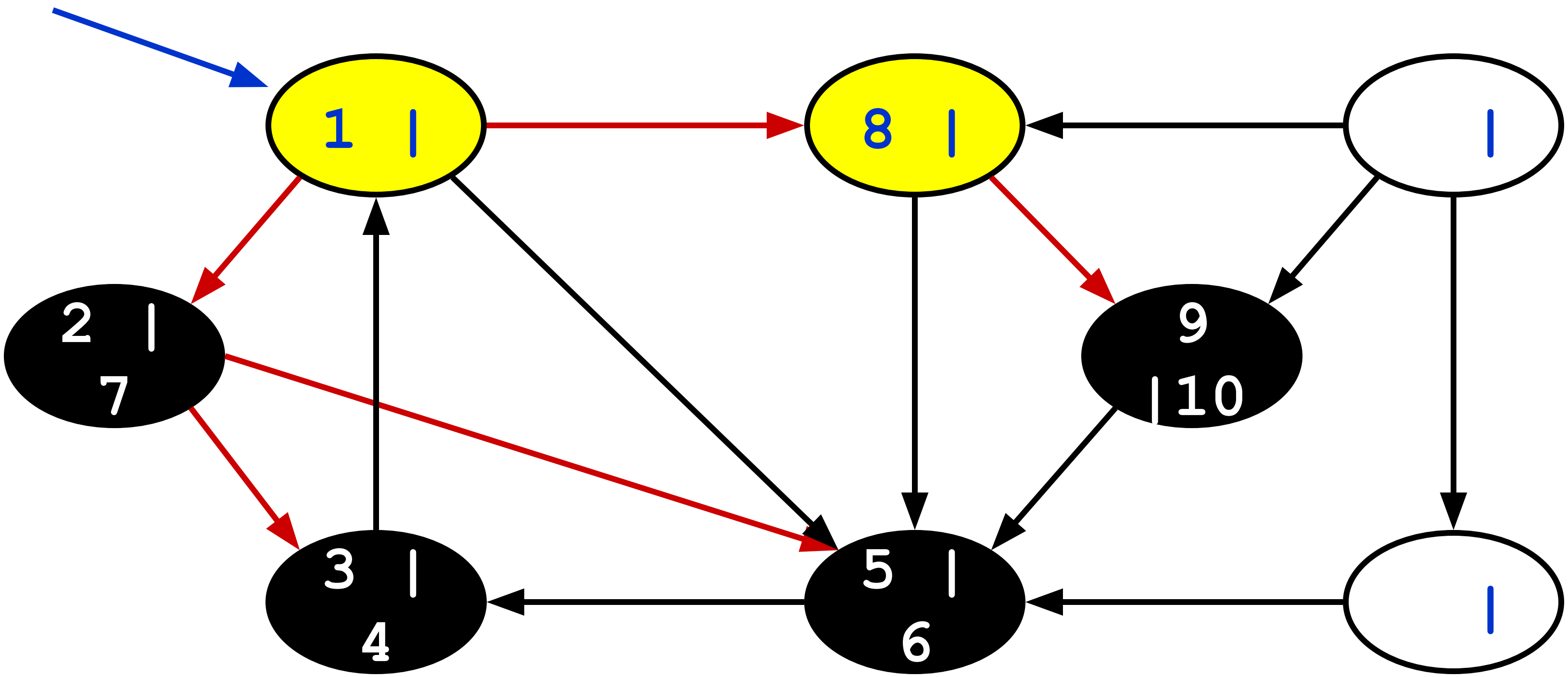


DFS Example

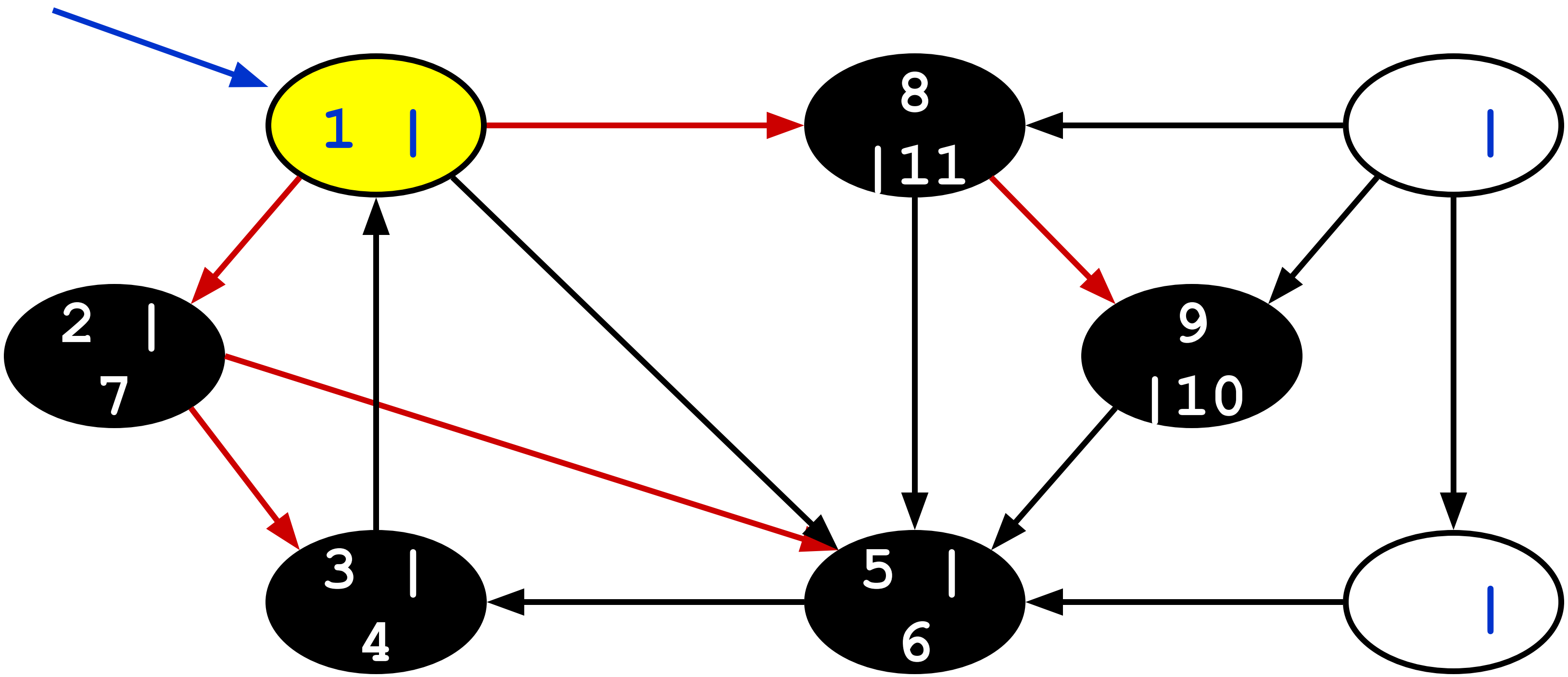


What is the structure of the yellow vertices?
What do they represent?

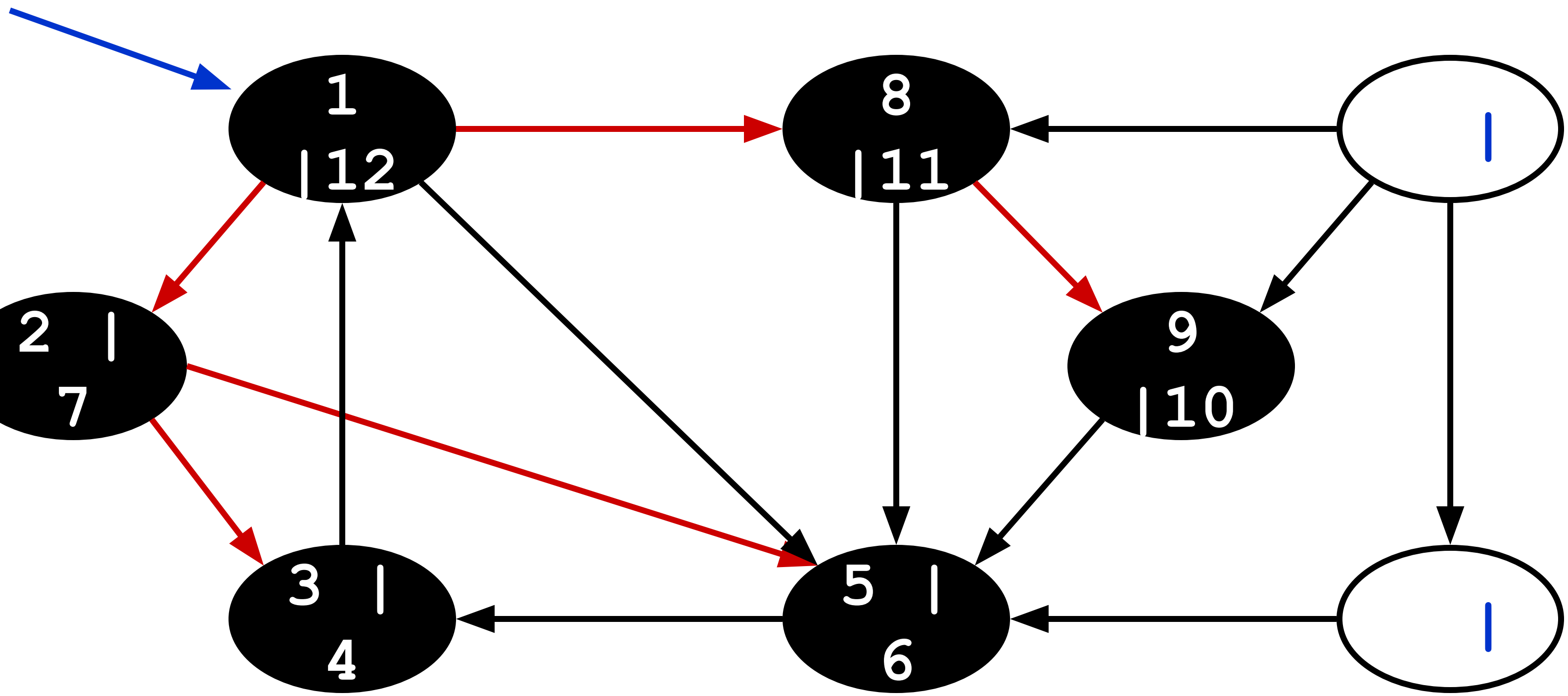
DFS Example



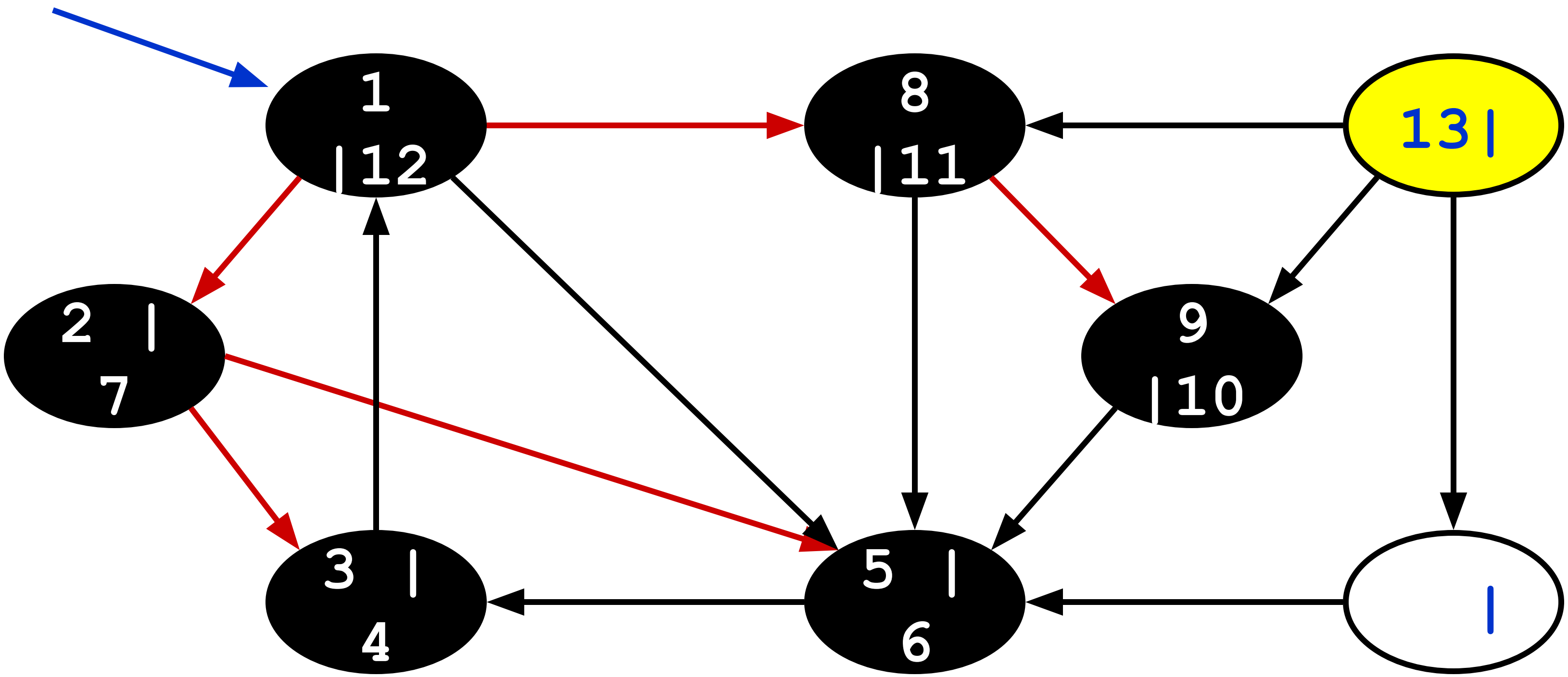
DFS Example



DFS Example



DFS Example



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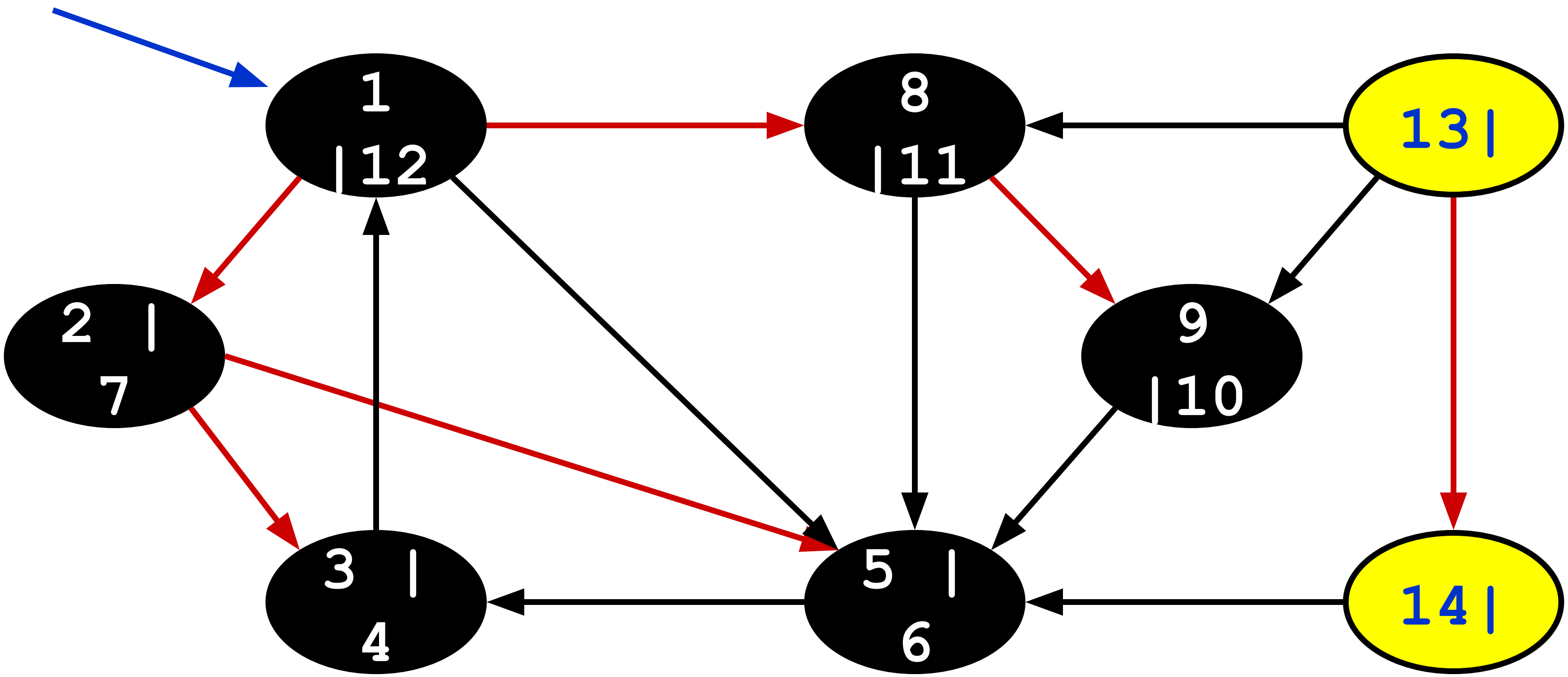
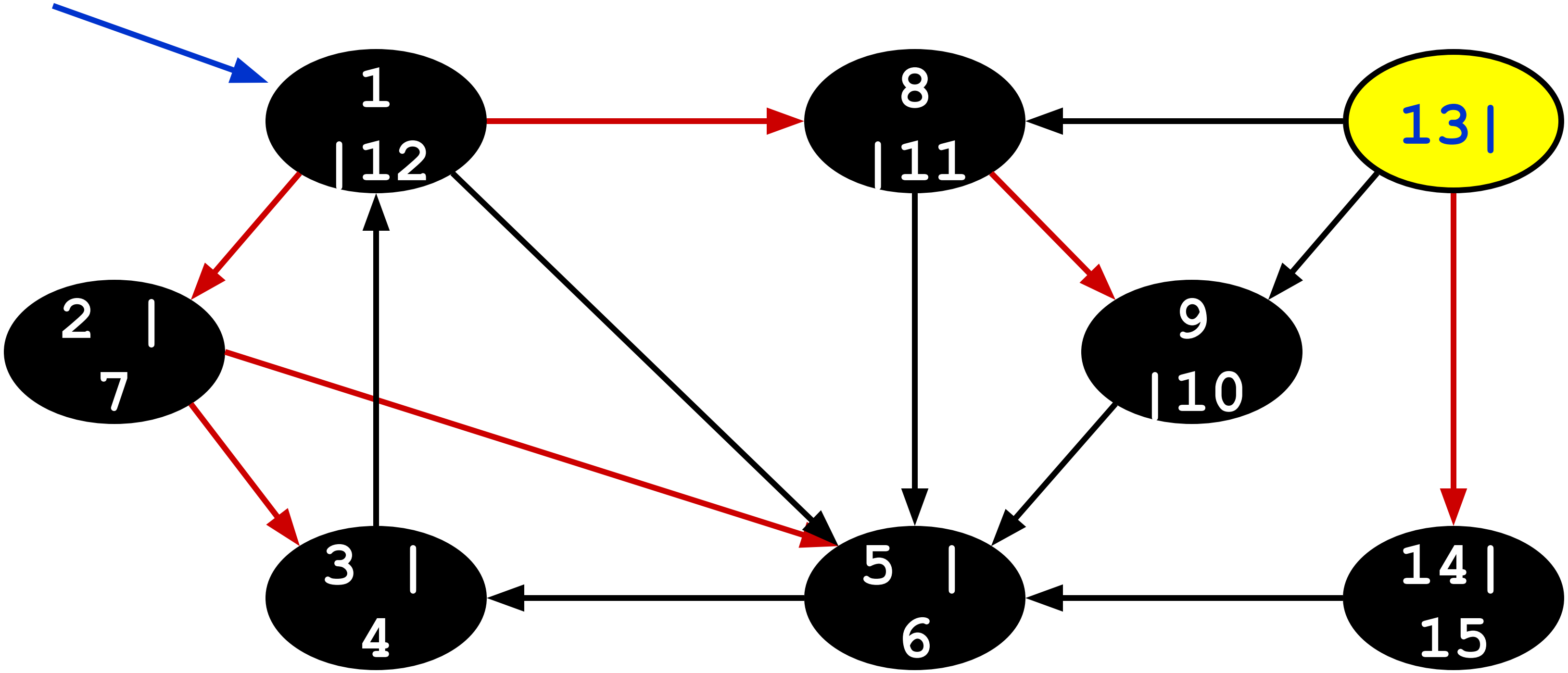
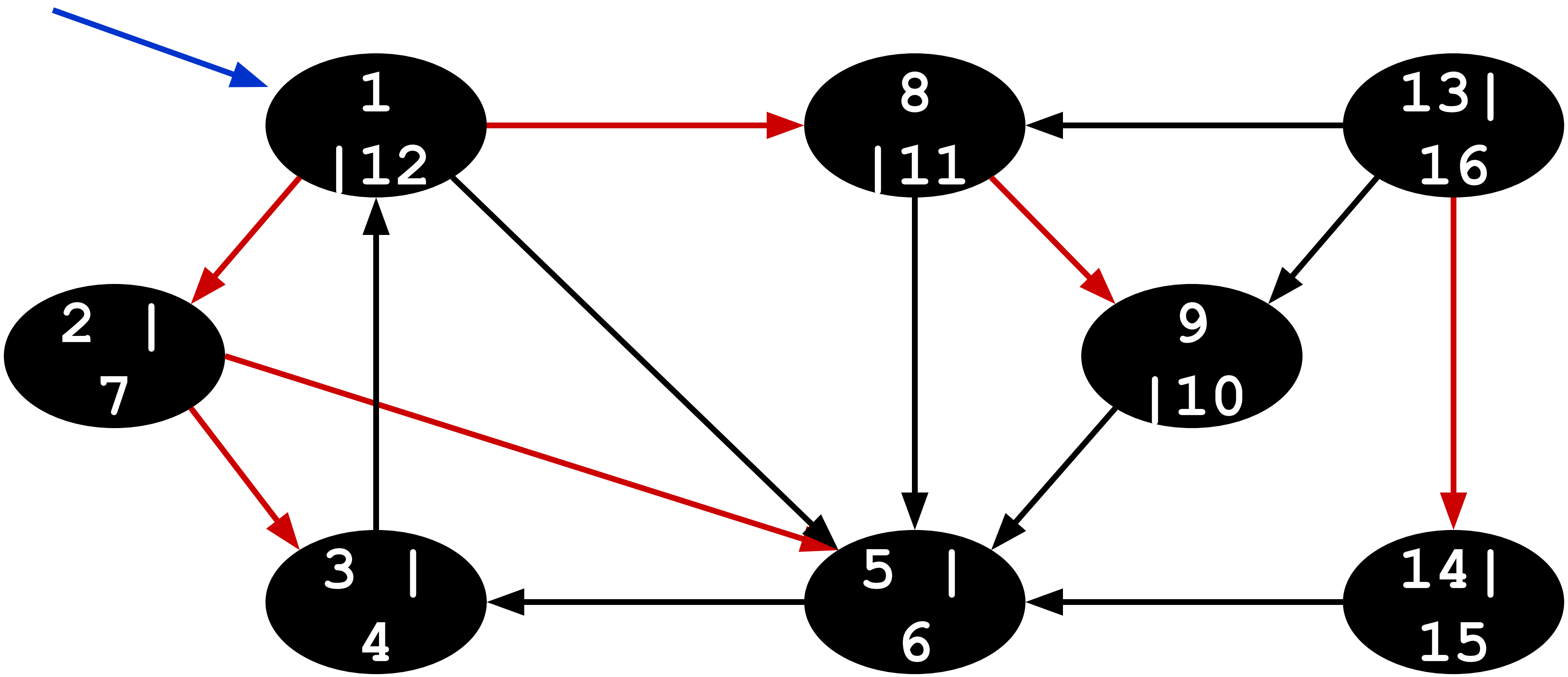


DFS Example



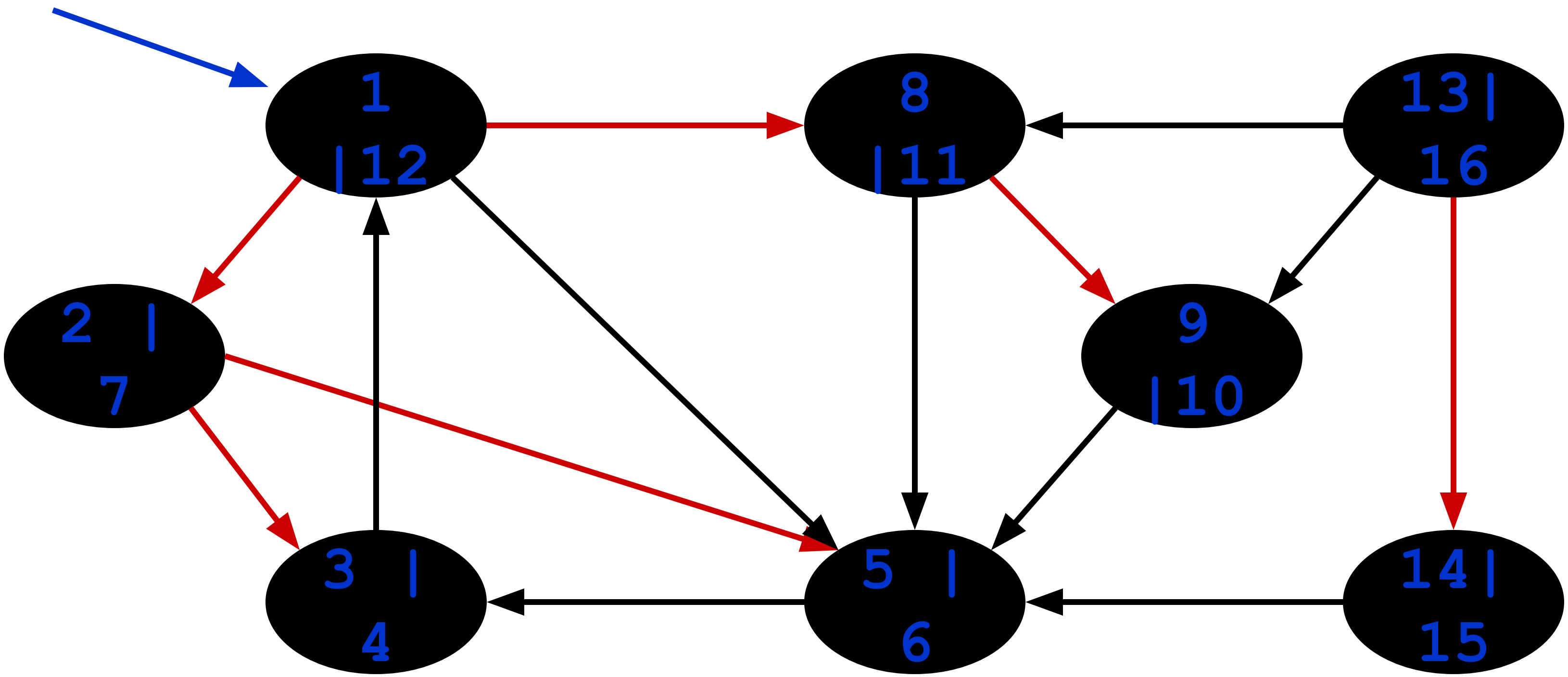
DFS Example



DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
    - The tree edges form a spanning forest
    - *Can tree edges form cycles? Why or why not?*

DFS Example

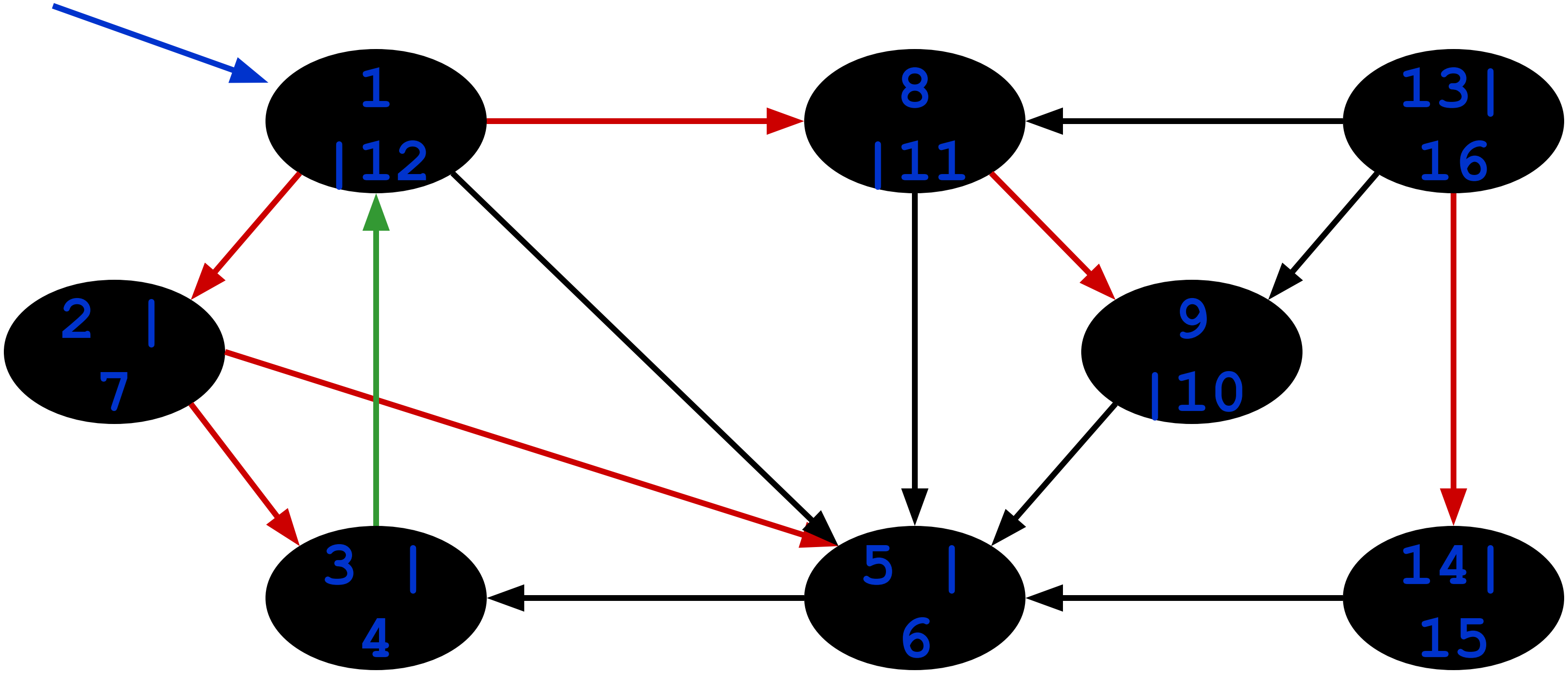


Tree edges

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
    - Encounter a yellow vertex (yellow to yellow)

DFS Example

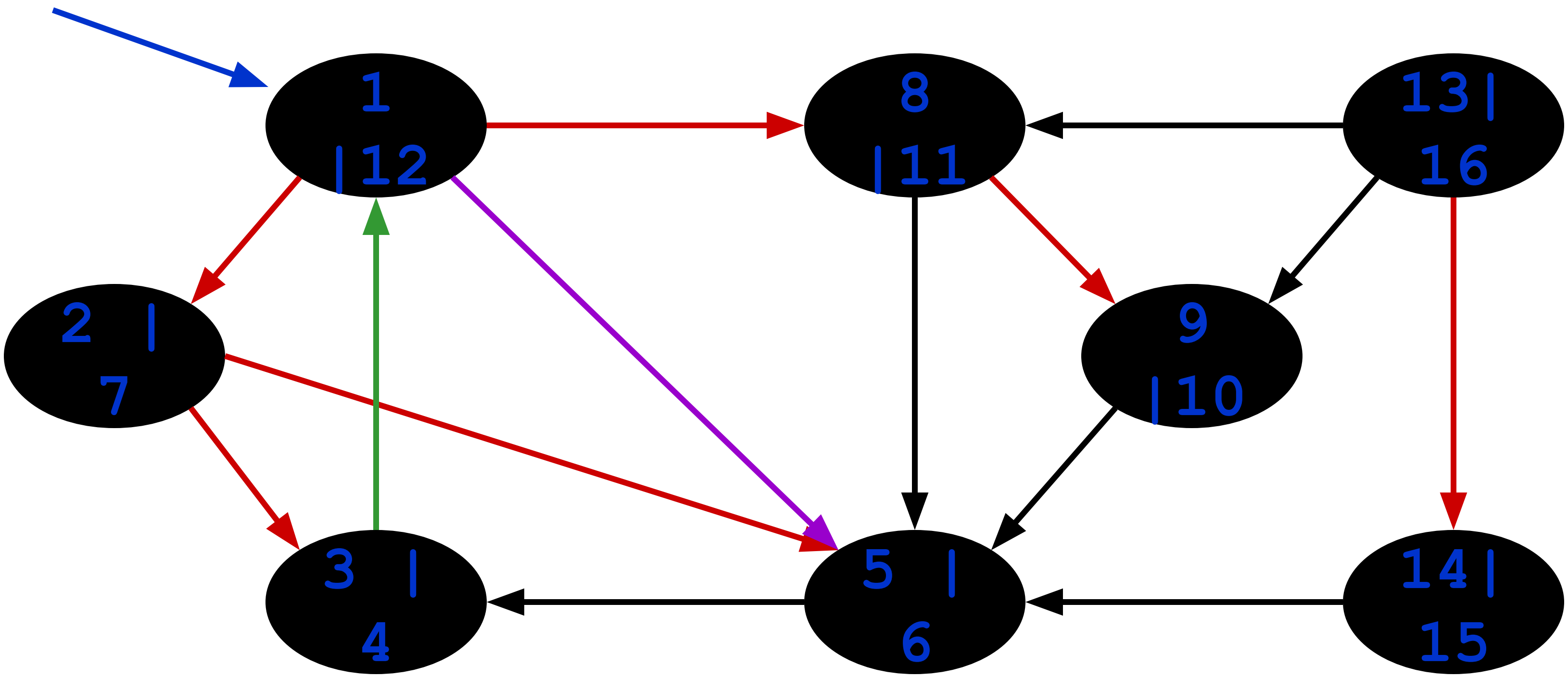


Tree edges ***Back edges***

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
    - Not a tree edge, though
    - From yellow node to black node

DFS Example

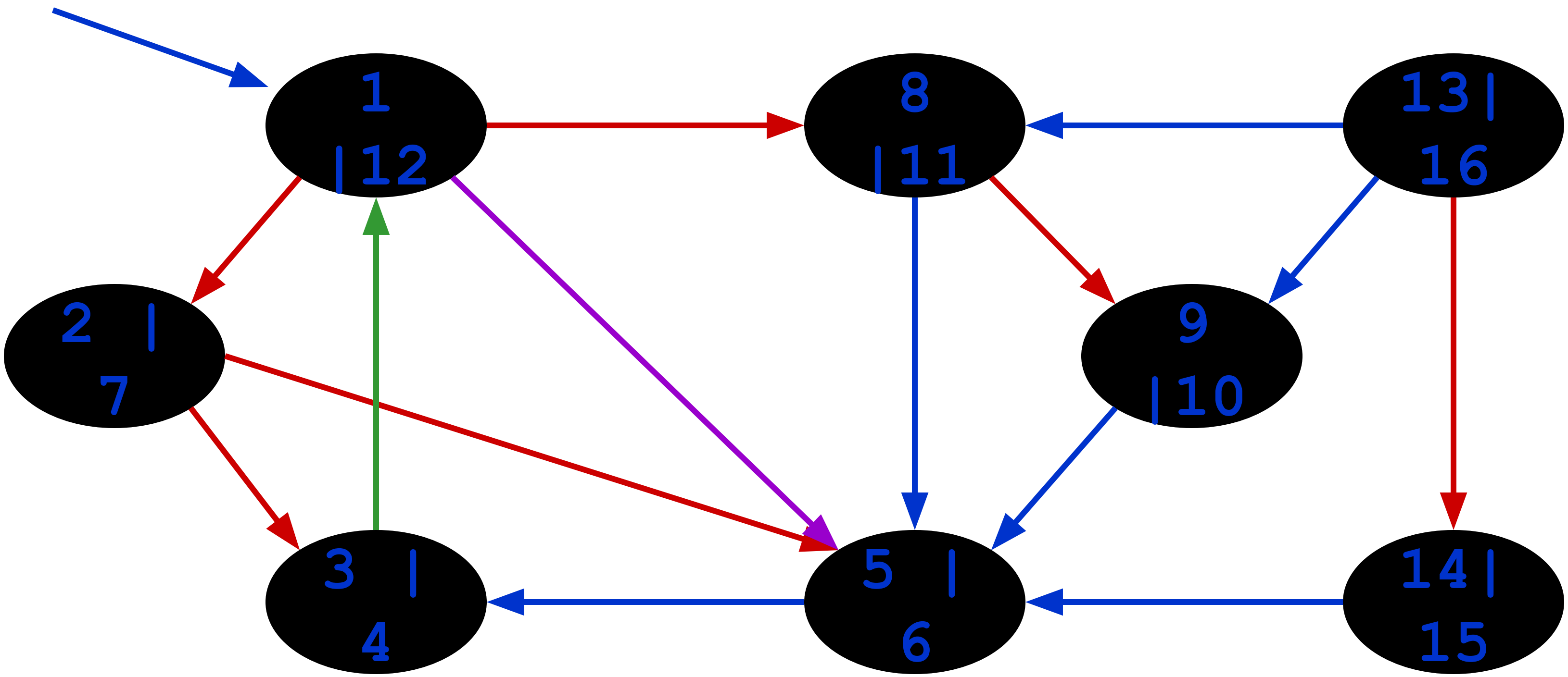


Tree edges *Back edges* *Forward edges*

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
  - *Cross edge*: between a tree or subtrees
    - From a yellow node to a black node

DFS Example



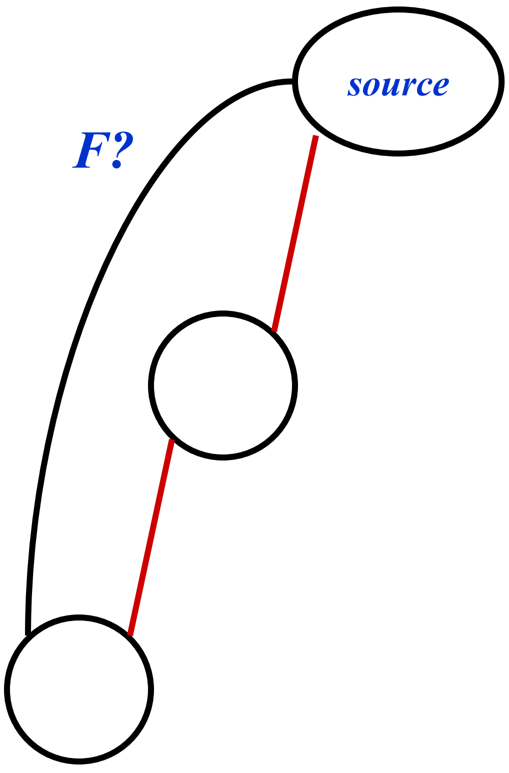
Tree edges *Back edges* *Forward edges* *Cross edges*

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
  - *Cross edge*: between a tree or subtrees
- Note: tree & back edges are important; most algorithms don't distinguish forward & cross

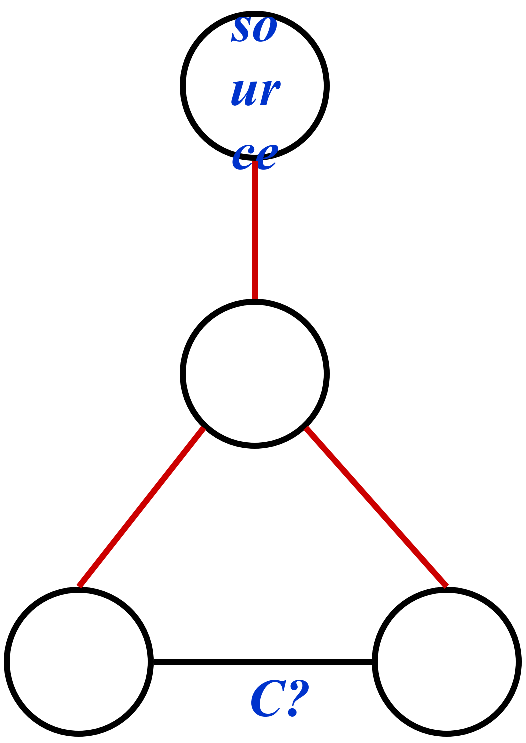
DFS: Kinds Of Edges

- Thm 23.9: If G is undirected, a DFS produces only tree and back edges
- Proof by contradiction:
  - Assume there's a forward edge
    - But F? edge must actually be a back edge (*why?*)



DFS: Kinds Of Edges

- Thm 23.9: If G is undirected, a DFS produces only tree and back edges
- Proof by contradiction:
  - Assume there's a cross edge
    - But C? edge cannot be cross:
    - must be explored from one of the vertices it connects, becoming a tree vertex, before other vertex is explored
    - So in fact the picture is wrong…both lower tree edges cannot in fact be tree edges



DFS And Graph Cycles

- Thm: An undirected graph is *acyclic* iff a DFS yields no back edges
  - If acyclic, no back edges (because a back edge implies a cycle
  - If no back edges, acyclic
    - No back edges implies only tree edges (*Why?*)
    - Only tree edges implies we have a tree or a forest
    - Which by definition is acyclic
- Thus, can run DFS to find whether a graph has a cycle

DFS And Cycles

- *How would you modify the code to detect cycles?*

```
DFS(G)
{
    for each vertex u ∈ G->V
    {
        u->color = WHITE;
    }
    time = 0;
    for each vertex u ∈ G->V
    {
        if (u->color == WHITE)
            DFS_Visit(u);
    }
}
```

```
DFS_Visit(u)
{
    u->color = GREY;
    time = time+1;
    u->d = time;
    for each v ∈ u->Adj[]
    {
        if (v->color == WHITE)
            DFS_Visit(v);
    }
    u->color = BLACK;
    time = time+1;
    u->f = time;
}
```

DFS And Cycles

- *What will be the running time?*

```
DFS(G)
{
    for each vertex u ∈ G->V
    {
        u->color = WHITE;
    }
    time = 0;
    for each vertex u ∈ G->V
    {
        if (u->color == WHITE)
            DFS_Visit(u);
    }
}
```

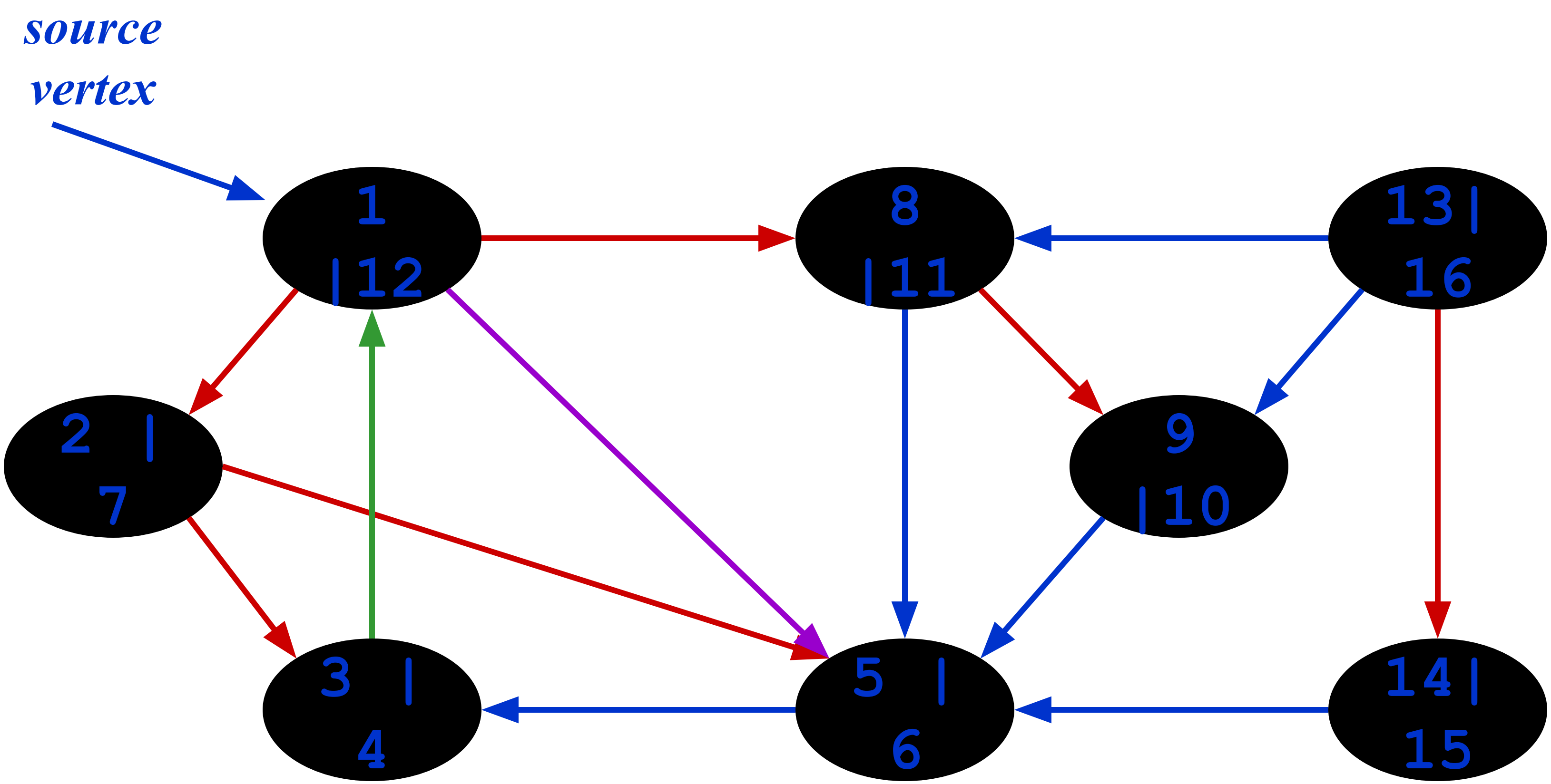
```
DFS_Visit(u)
{
    u->color = GREY;
    time = time+1;
    u->d = time;
    for each v ∈ u->Adj[]
    {
        if (v->color == WHITE)
            DFS_Visit(v);
    }
    u->color = BLACK;
    time = time+1;
    u->f = time;
}
```

DFS And Cycles

- *What will be the running time?*
- A: O(V+E)
- We can actually determine if cycles exist in O(V) time:
  - In an undirected acyclic forest, $|E| \leq |V| - 1$
  - So count the edges: if ever see $|V|$ distinct edges, must have seen a back edge along the way

Topological Sort
Minimum Spanning Trees

Review: Kinds of Edges



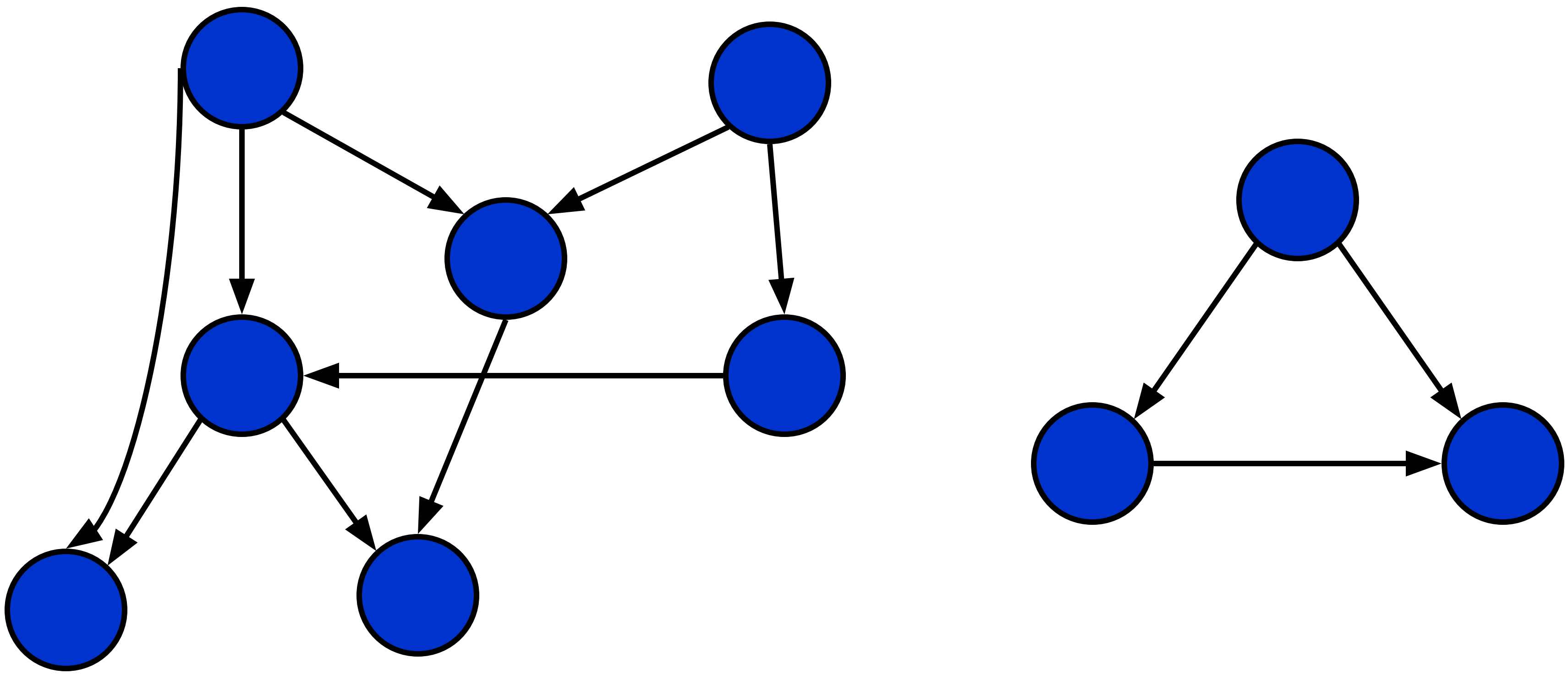
Tree edges *Back edges* *Forward edges* *Cross edges*

DFS And Cycles

- Running time: O(V+E)
- We can actually determine if cycles exist in O(V) time:
  - In an undirected acyclic forest, $|E| \leq |V| - 1$
  - So count the edges: if ever see |V| distinct edges, must have seen a back edge along the way
  - *Why not just test if $|E| < |V|$ and answer the question in constant time?*

Directed Acyclic Graphs

- A *directed acyclic graph* or *DAG* is a directed graph with no directed cycles:



David Luebke

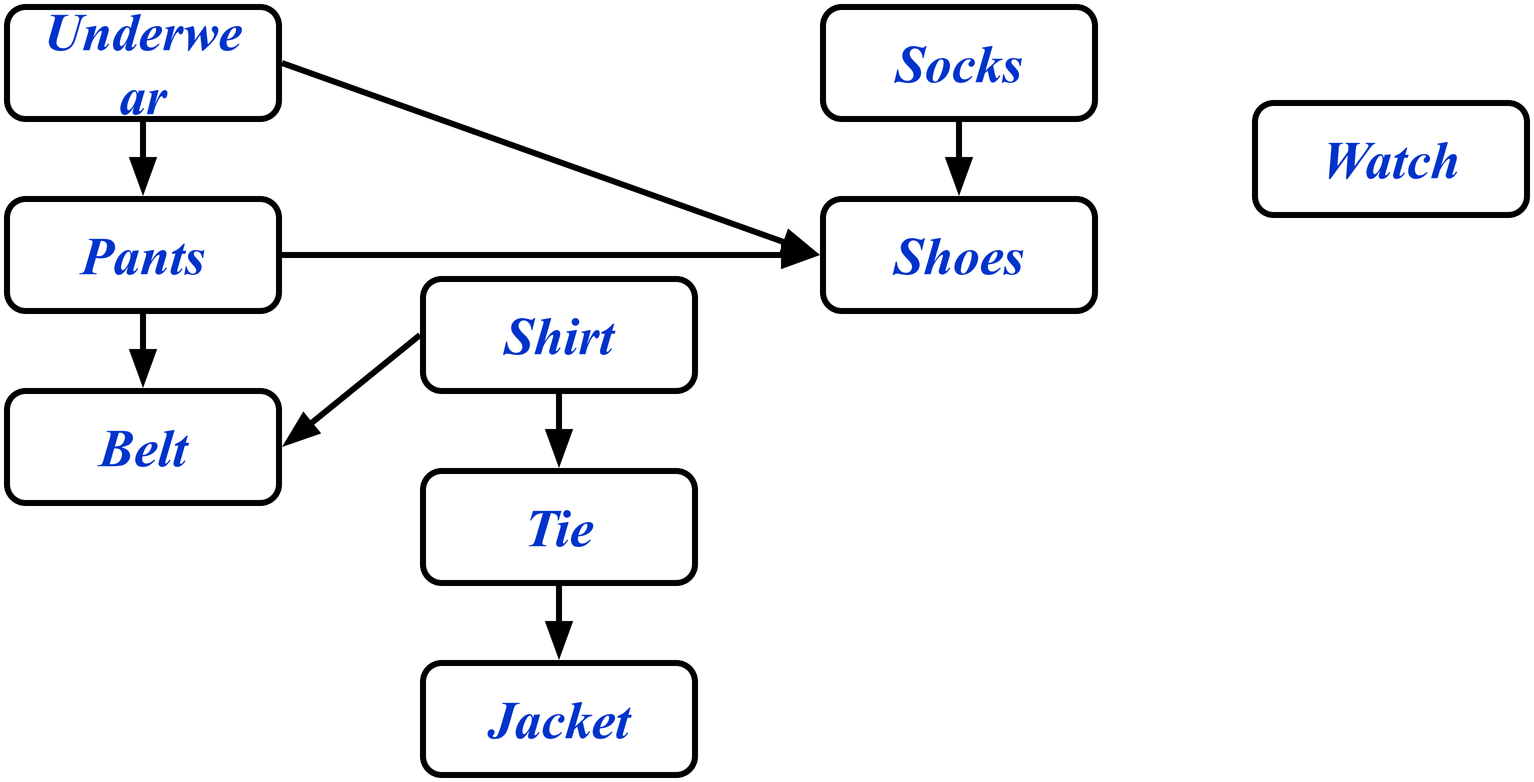
DFS and DAGs

- Argue that a directed graph G is acyclic iff a DFS of G yields no back edges:
  - Forward: if G is acyclic, will be no back edges
    - Trivial: a back edge implies a cycle
  - Backward: if no back edges, G is acyclic
    - Argue contrapositive: G has a cycle $\Rightarrow$ $\exists$ a back edge
      - Let v be the vertex on the cycle first discovered, and u be the predecessor of v on the cycle
      - When v discovered, whole cycle is white
      - Must visit everything reachable from v before returning from DFS-Visit()
      - So path from u→v is yellow→yellow, thus (u, v) is a back edge

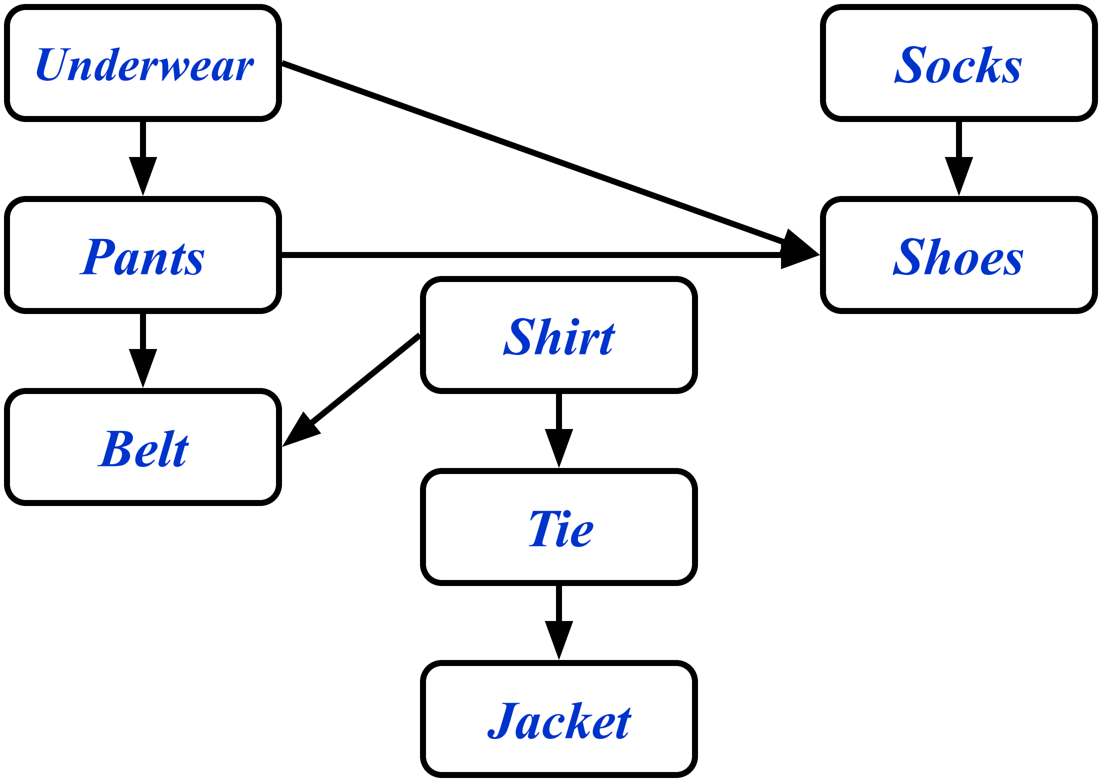
Topological Sort

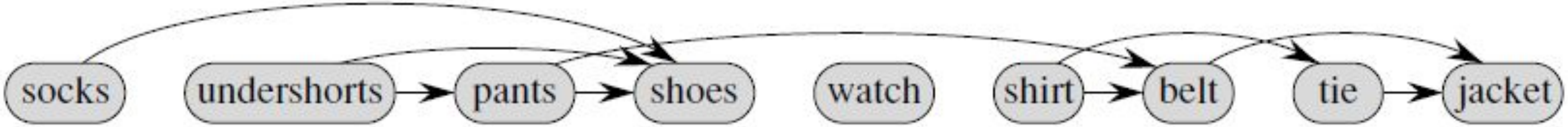
- *Topological sort* of a DAG:
  - Linear ordering of all vertices in graph G such that vertex u comes before vertex v if edge $(u, v) \in G$
- Real-world example: getting dressed

Getting Dressed



Getting Dressed





Topological Sort Algorithm

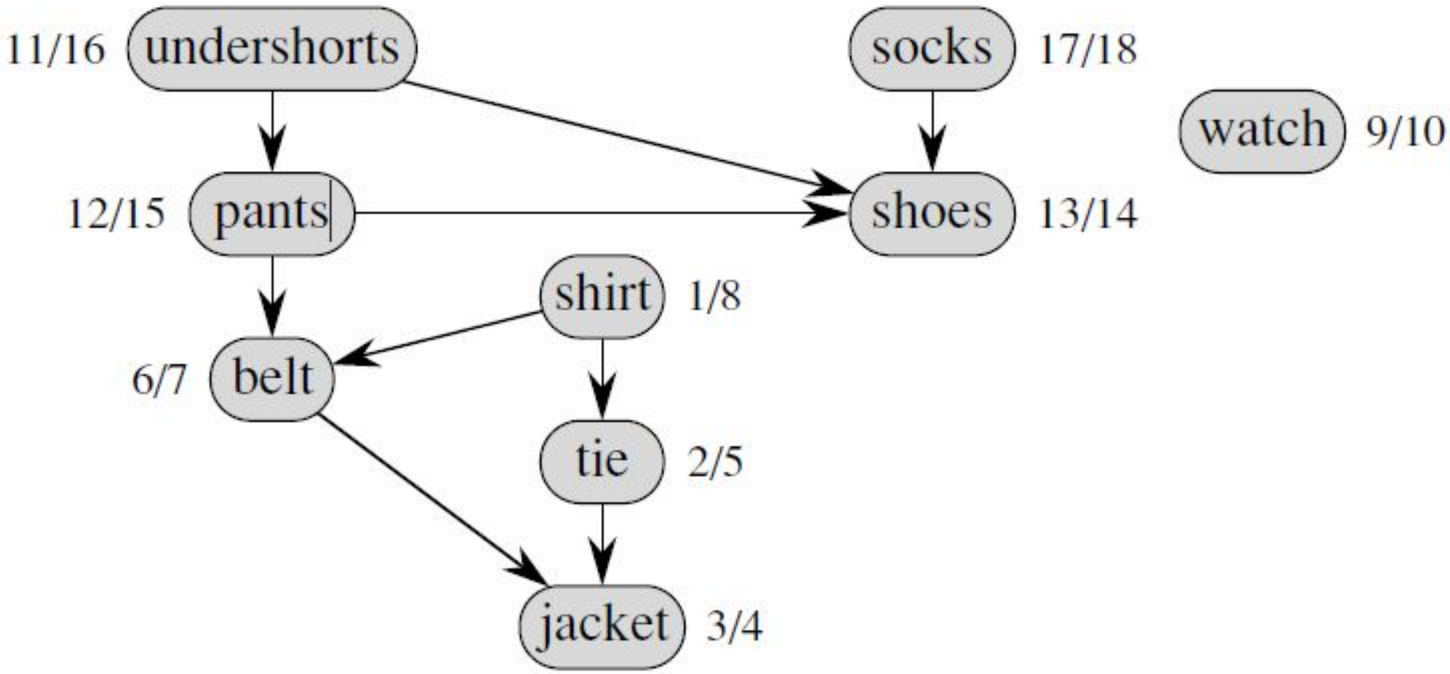
```
Topological-Sort()
{
   Run DFS
   When a vertex is finished, output it
   Vertices are output in reverse
     topological order
}
```

- Time: O(V+E)
- Correctness: Want to prove that
  $(u,v) \in G \Rightarrow u{\rightarrow}f > v{\rightarrow}f$

Topological Sort Algorithm

TOPOLOGICAL-SORT(G)

```
call DFS(G) to compute finishing times f[v] for each vertex v
as each vertex is finished, insert it onto the front of a linked list
return the linked list of vertices
```


11/16
undershorts
socks
17/18
watch
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pants
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shirt
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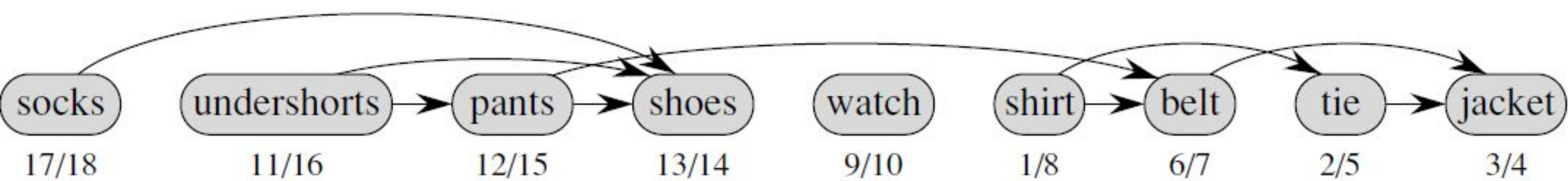
tie
2/5
jacket
3/4


socks
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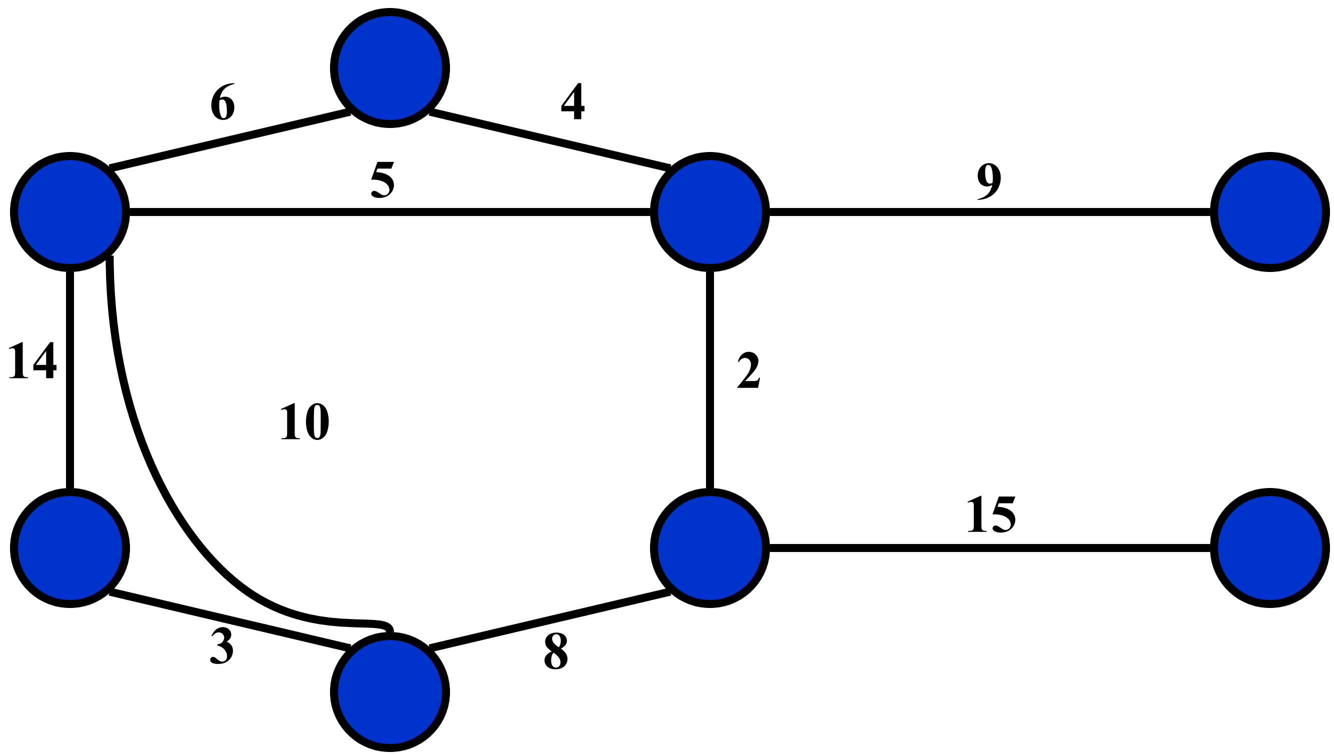
tie
jacket
17/18
11/16
12/15
13/14
9/10
1/8
6/7
2/5
3/4

Correctness of Topological Sort

- Claim: $(u,v) \in G \Rightarrow u{\rightarrow}f > v{\rightarrow}f$
  - When (u,v) is explored, u is yellow
    - v = yellow $\Rightarrow$ (u,v) is back edge. Contradiction (*Why?*)
    - v = white $\Rightarrow$ v becomes descendent of u $\Rightarrow$ $v{\rightarrow}f < u{\rightarrow}f$ (since must finish v before backtracking and finishing u)
    - v = black $\Rightarrow$ v already finished $\Rightarrow$ $v{\rightarrow}f < u{\rightarrow}f$

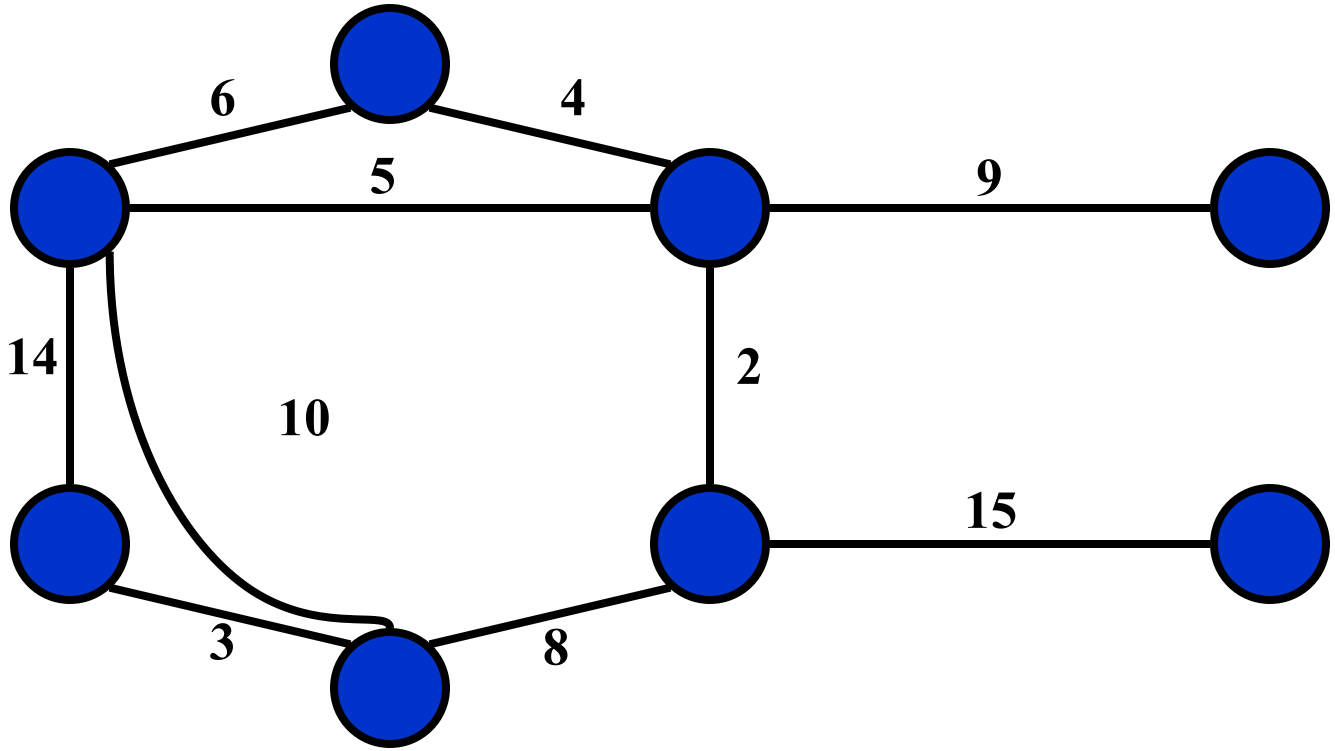
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph:



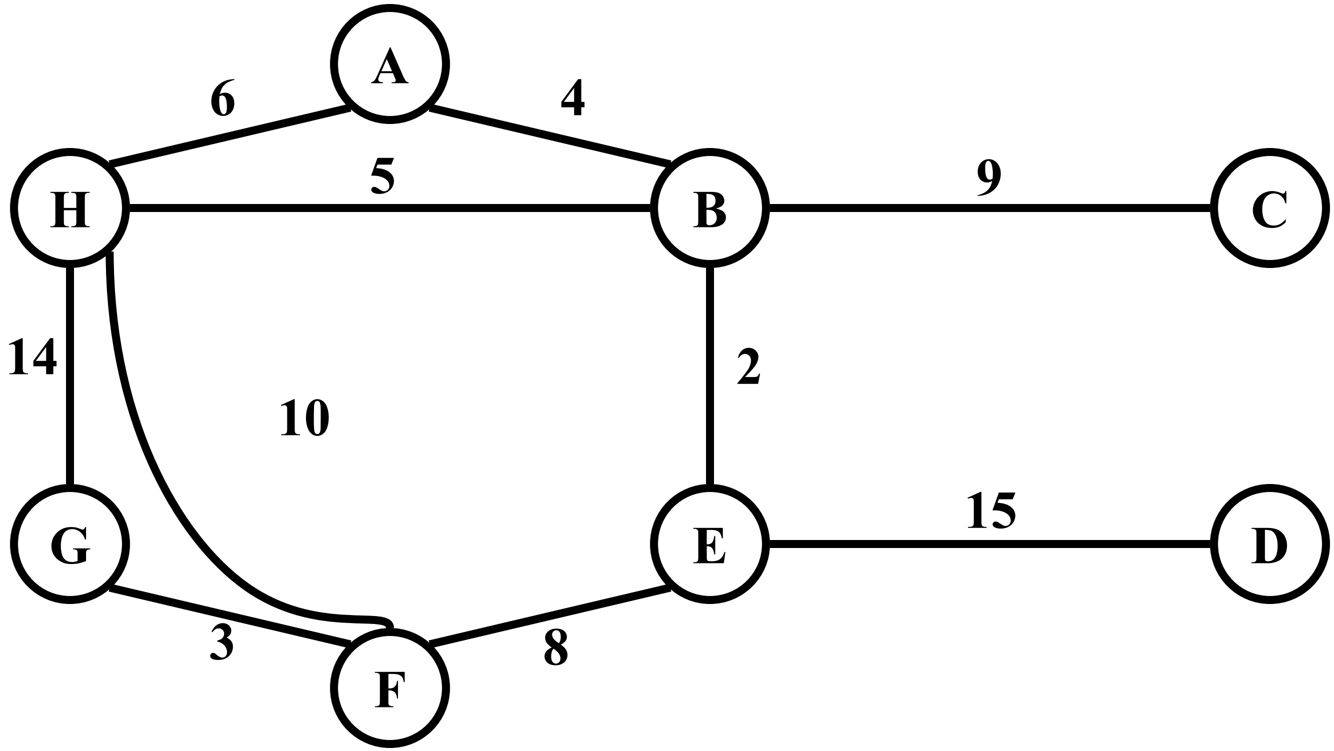
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph, find a *spanning tree* using edges that minimize the total weight



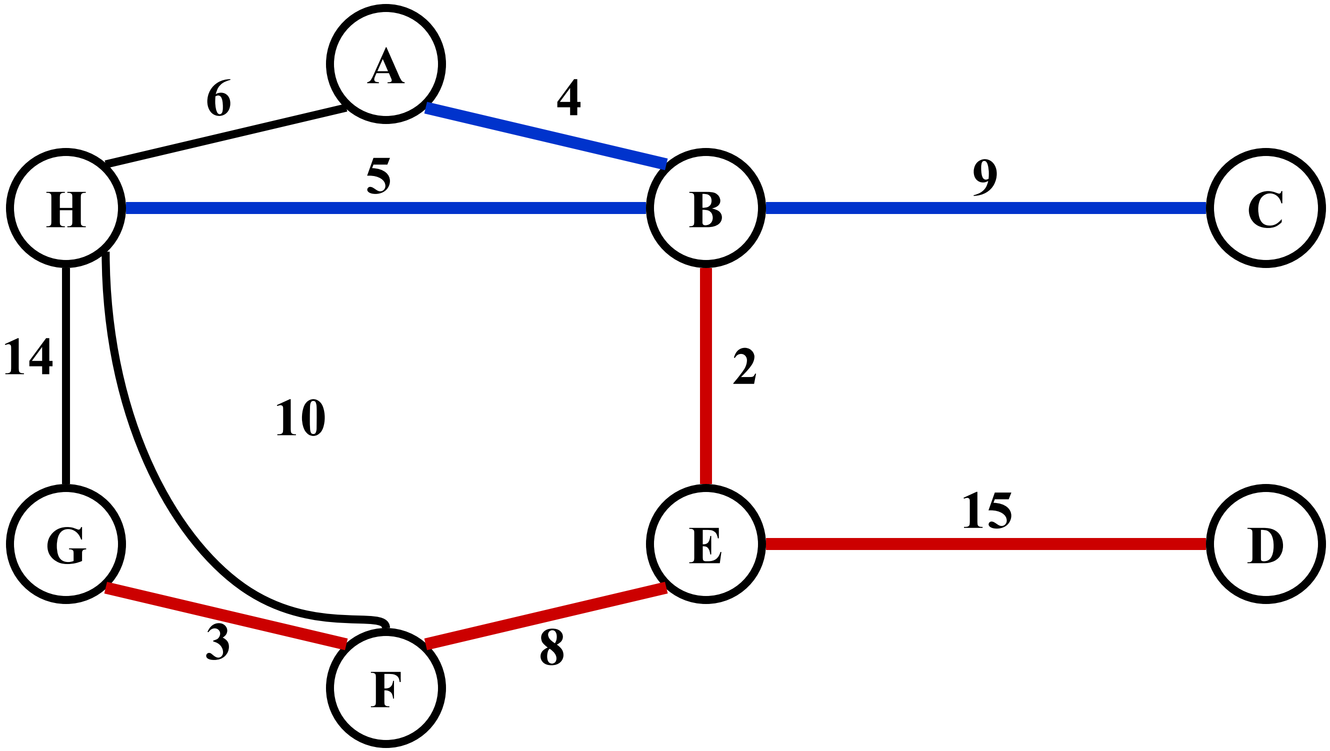
Minimum Spanning Tree

- *Which edges form the minimum spanning tree (MST) of the below graph?*

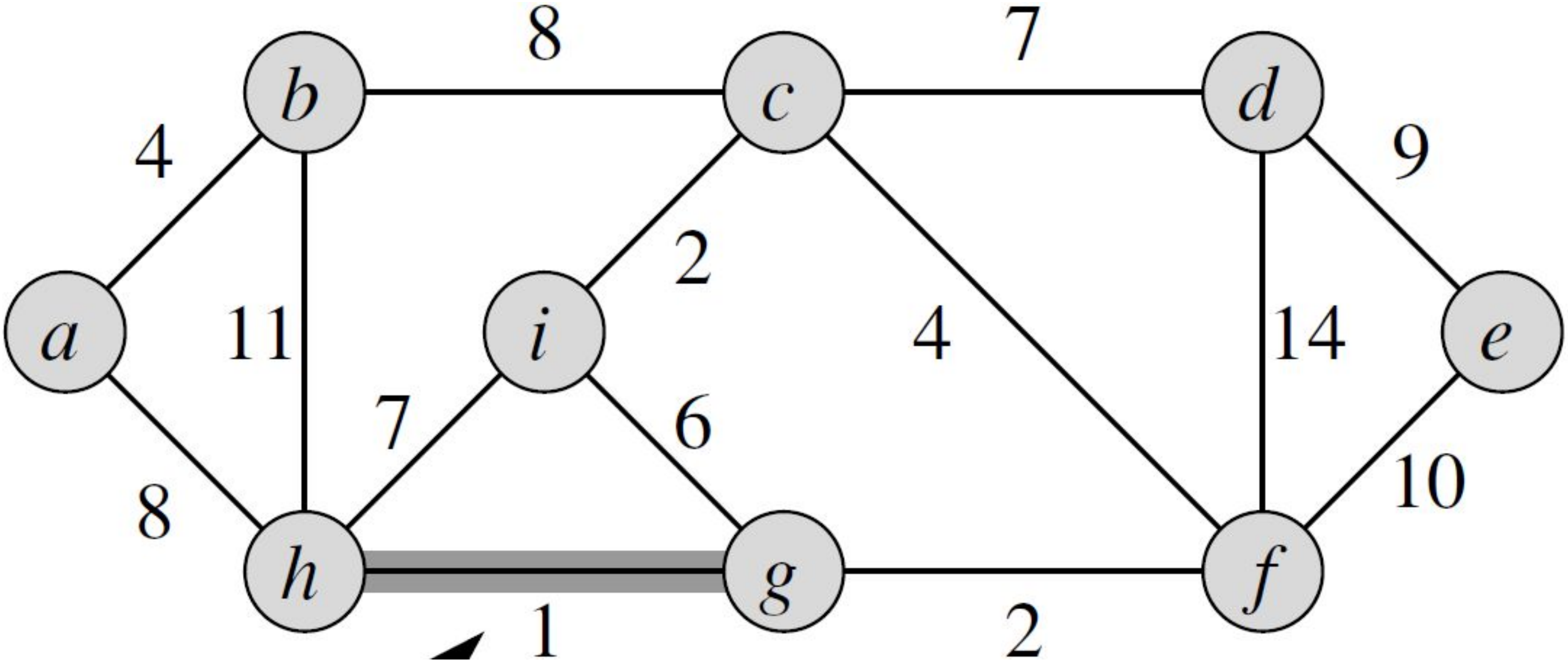


Minimum Spanning Tree

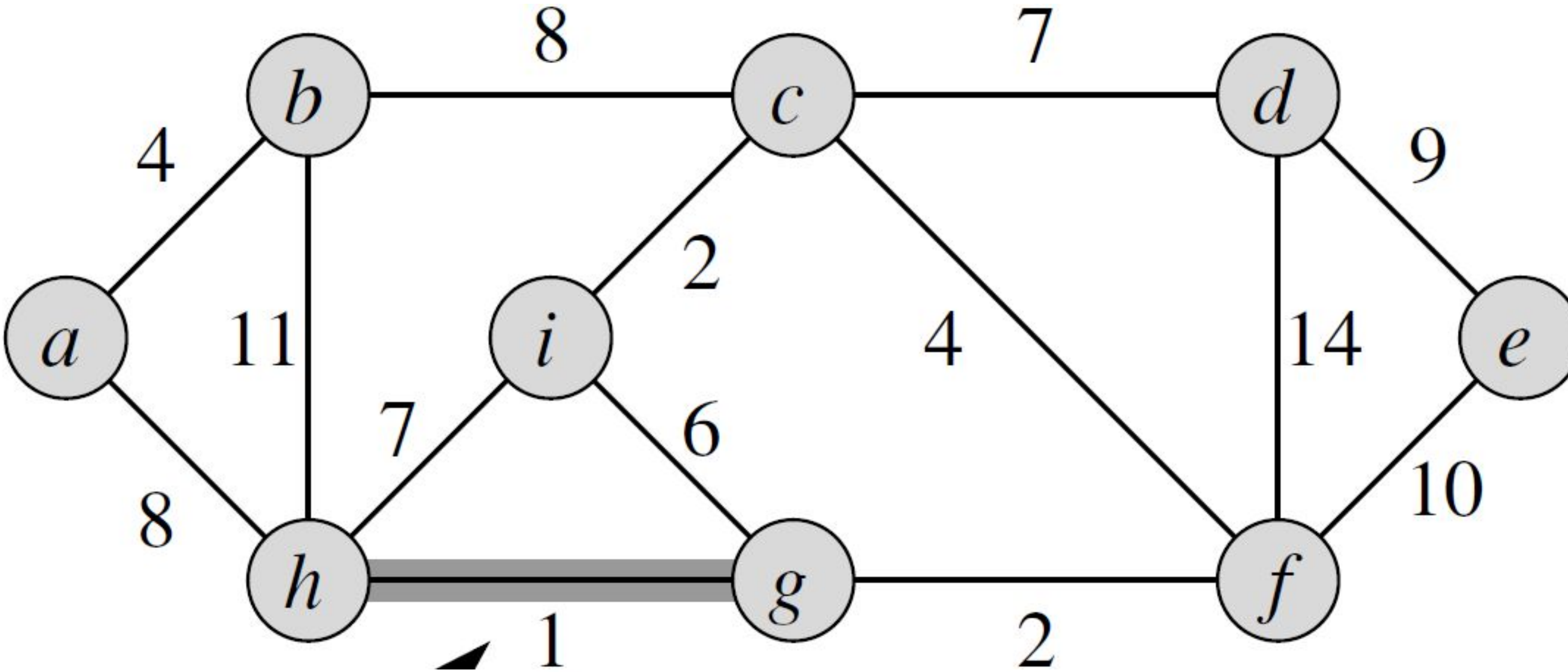
- Answer:



Another MST



Another MST



Minimum Spanning Tree

- MSTs satisfy the *optimal substructure* property: an optimal tree is composed of optimal subtrees
  - Let T be an MST of G with an edge (u,v) in the middle
  - Removing (u,v) partitions T into two trees T_1 and T_2
  - Claim: T_1 is an MST of $G_1 = (V_1,E_1)$, and T_2 is an MST of $G_2 = (V_2,E_2)$ (*Do V_1 and V_2 share vertices? Why?*)
  - Proof: $w(T) = w(u,v) + w(T_1) + w(T_2)$
    (There can't be a better tree than T_1 or T_2, or T would be suboptimal)

Minimum Spanning Tree

- Thm:
  - Let T be MST of G, and let A $\subseteq$ T be subtree of T
  - Let (u,v) be min-weight edge connecting A to V-A
  - Then $(u,v) \in$ T

Minimum Spanning Tree

- Thm:
  - Let T be MST of G, and let A $\subseteq$ T be subtree of T
  - Let (u,v) be min-weight edge connecting A to V-A
  - Then $(u,v) \in$ T
- Proof: in book (see Thm 24.1)

Prim’s Algorithm

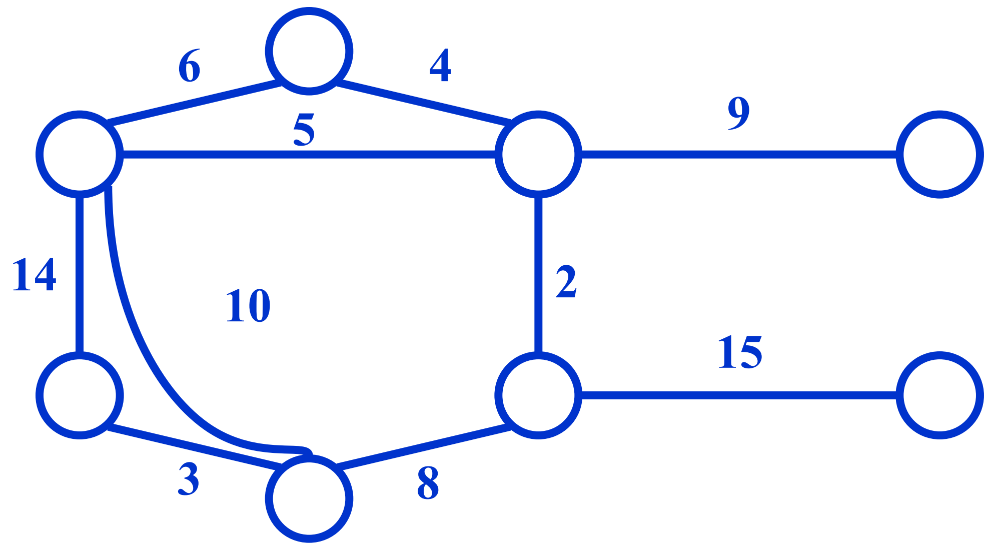
```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```

MST-Prim algorithm

```
MST-PRIM(G, w, r)
for each u ∈ V[G]
        do key[u] ← ∞
           π[u] ← NIL
key[r] ← 0
Q ← V[G]
while Q ≠ Ø
        do u ← EXTRACT-MIN(Q)
           for each v ∈ Adj[u]
               do if v ∈ Q and w(u, v) < key[v]
                     then π[v] ← u
                          key[v] ← w(u, v)
```

Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```



Run on example graph

Prim's Algorithm

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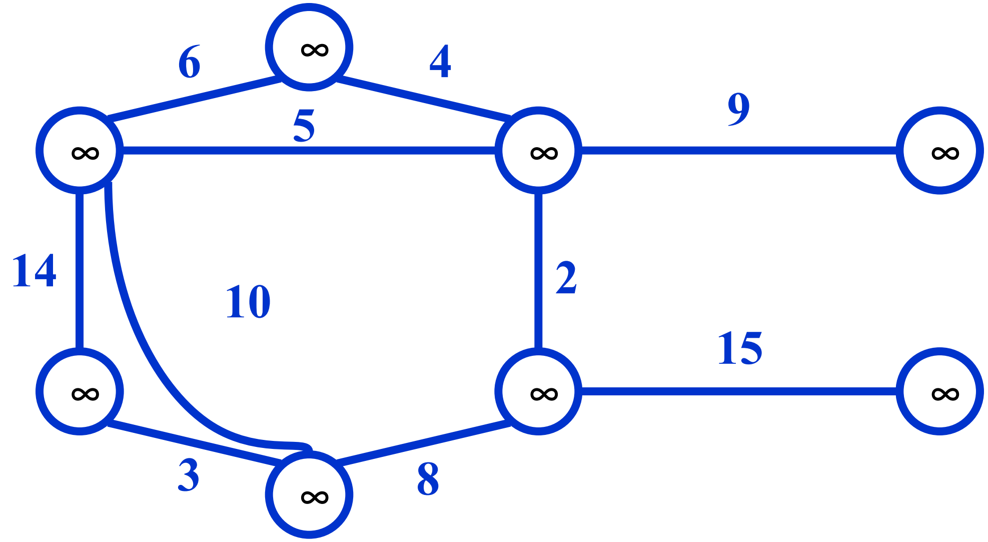
```



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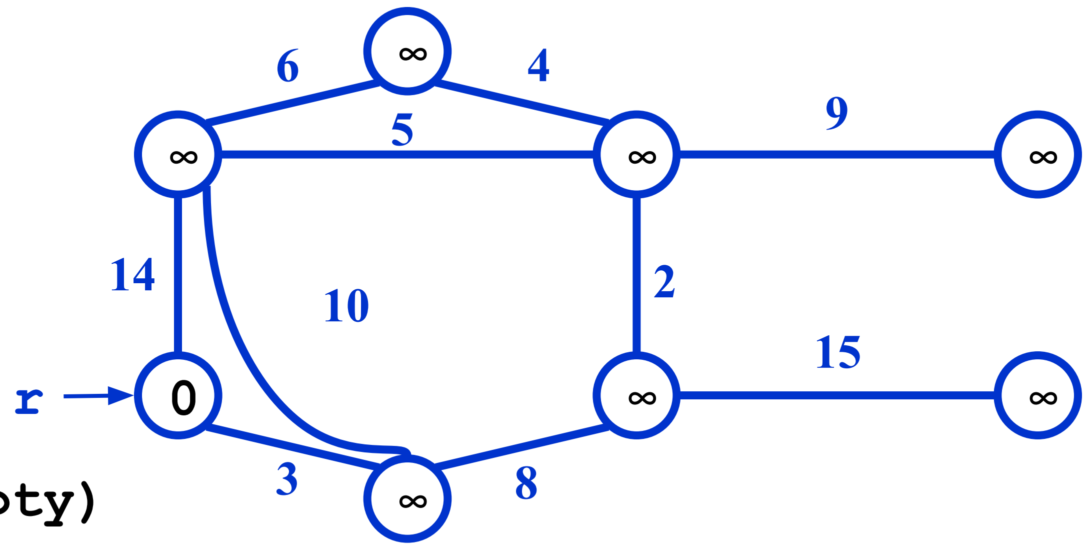

Pick a start vertex r

Prim's Algorithm

```
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    Q = V[G];
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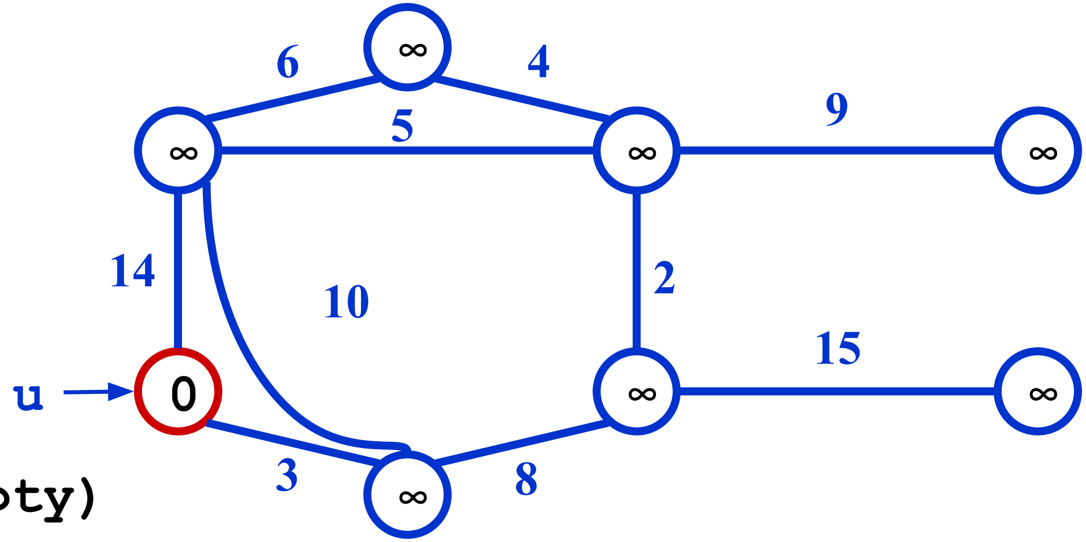
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```



Red vertices have been removed from Q

Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
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            if (v ∈ Q and w(u,v) < key[v])
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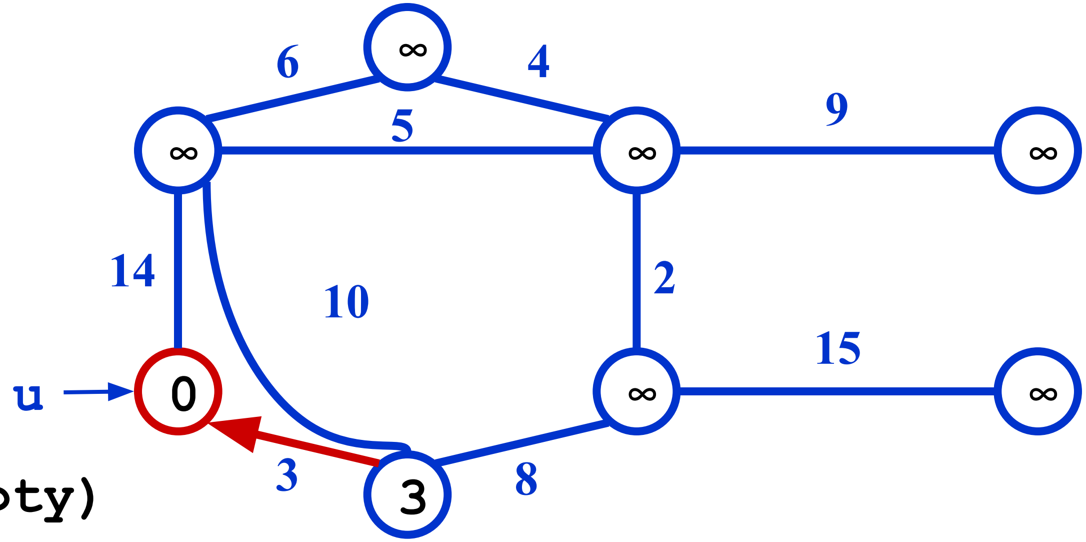
```



Red arrows indicate parent pointers

Prim's Algorithm

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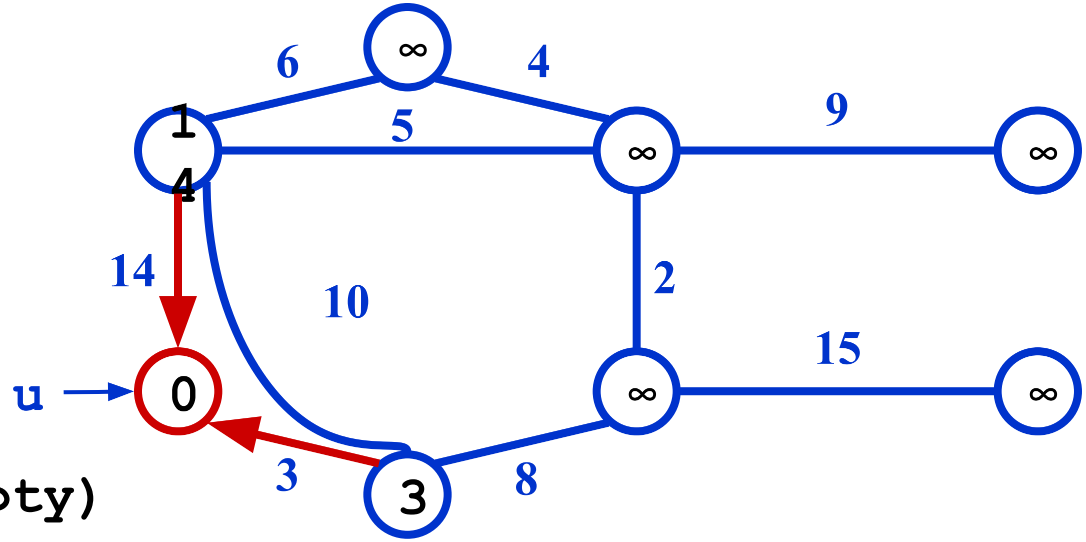

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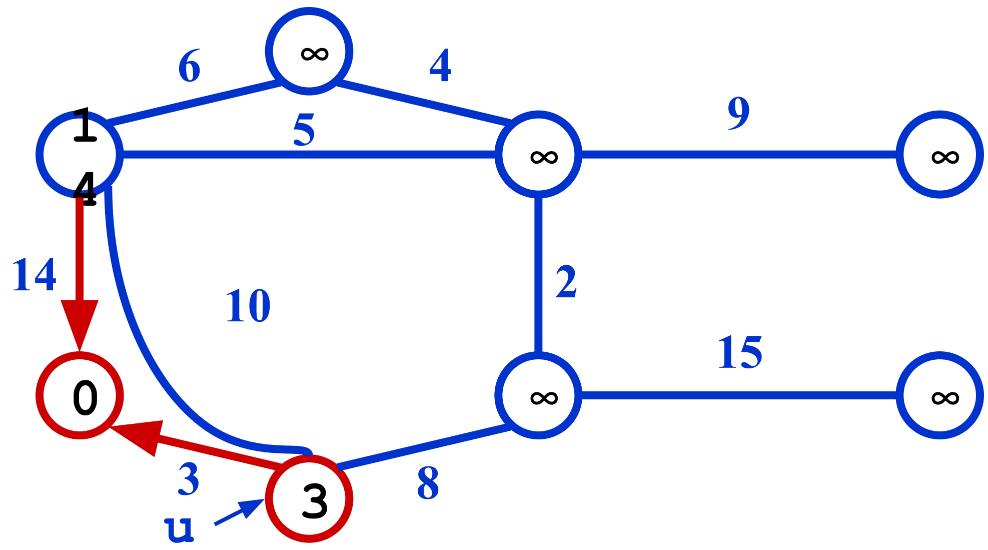

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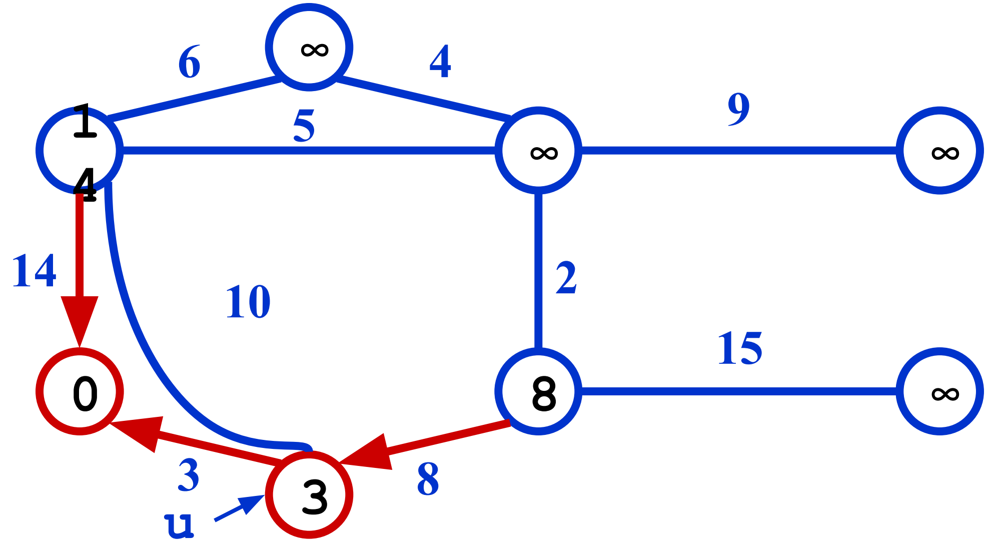

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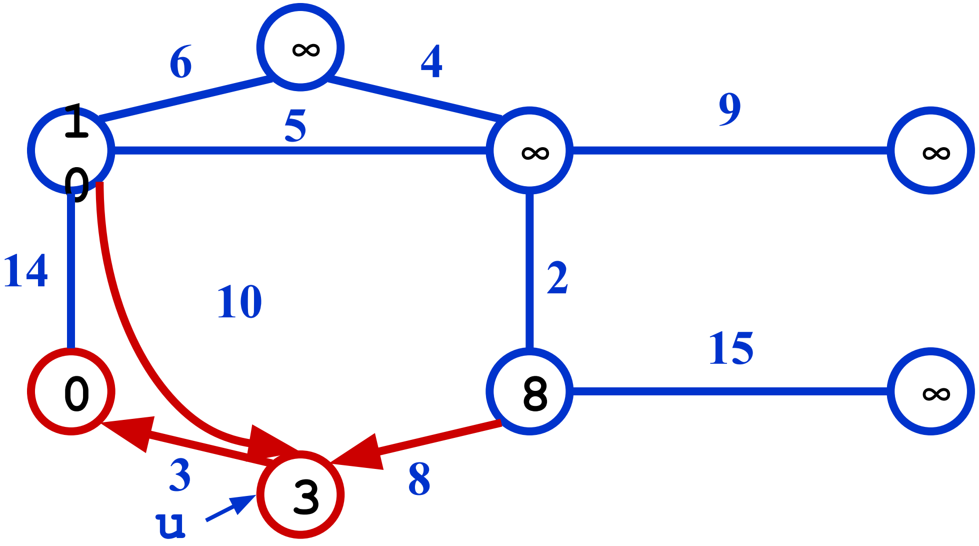

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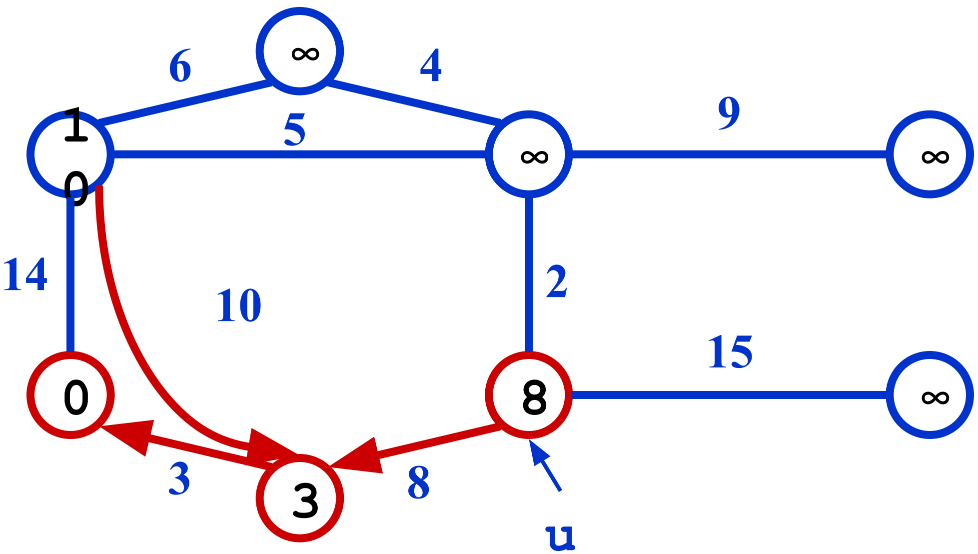

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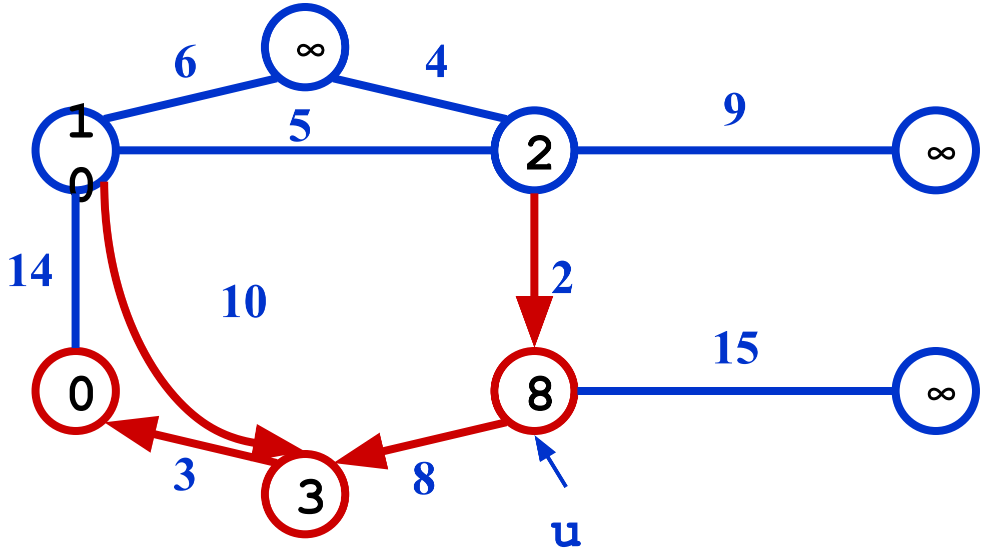

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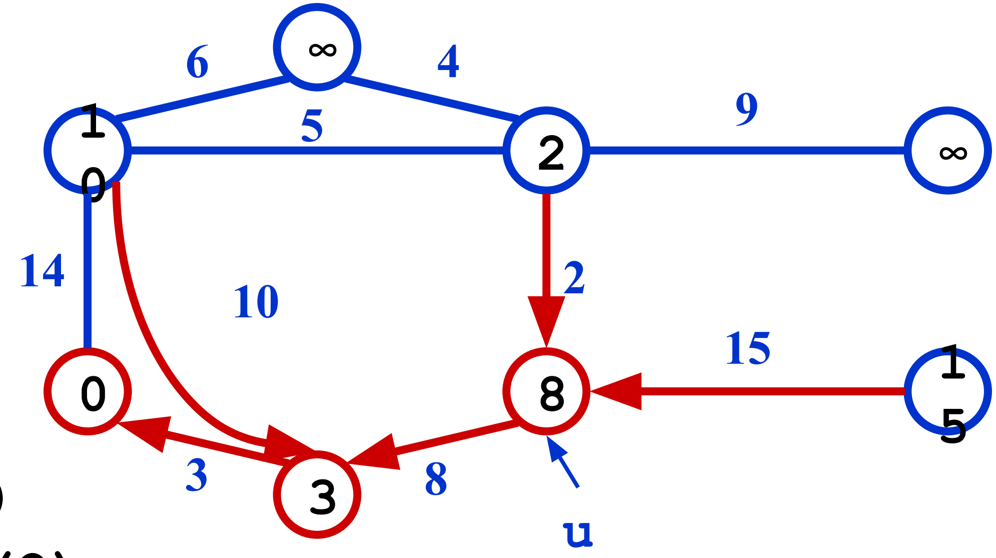

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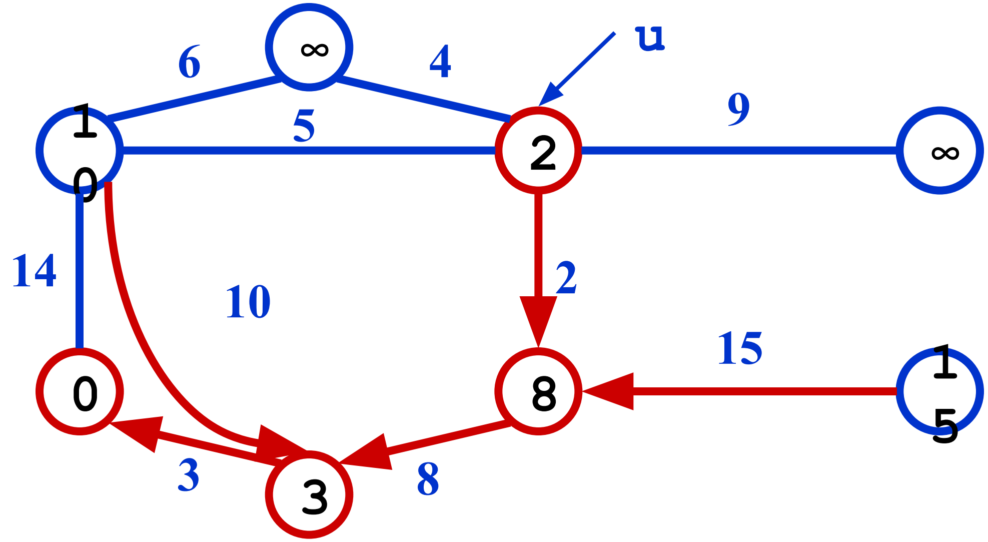

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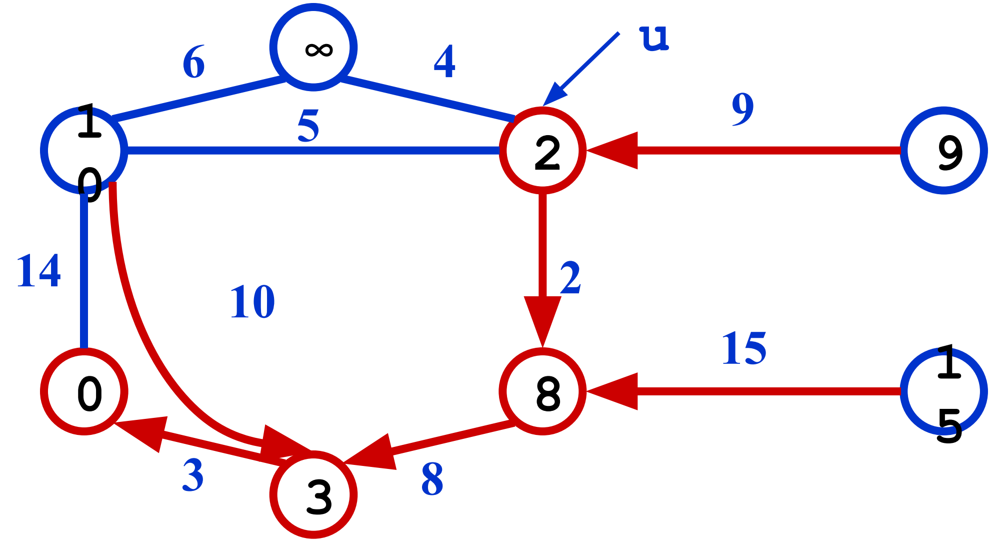

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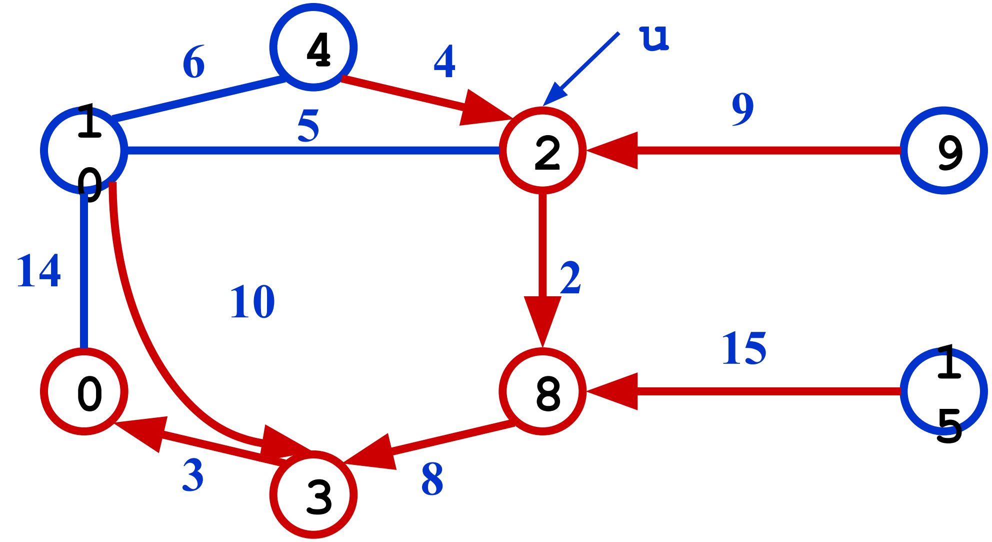

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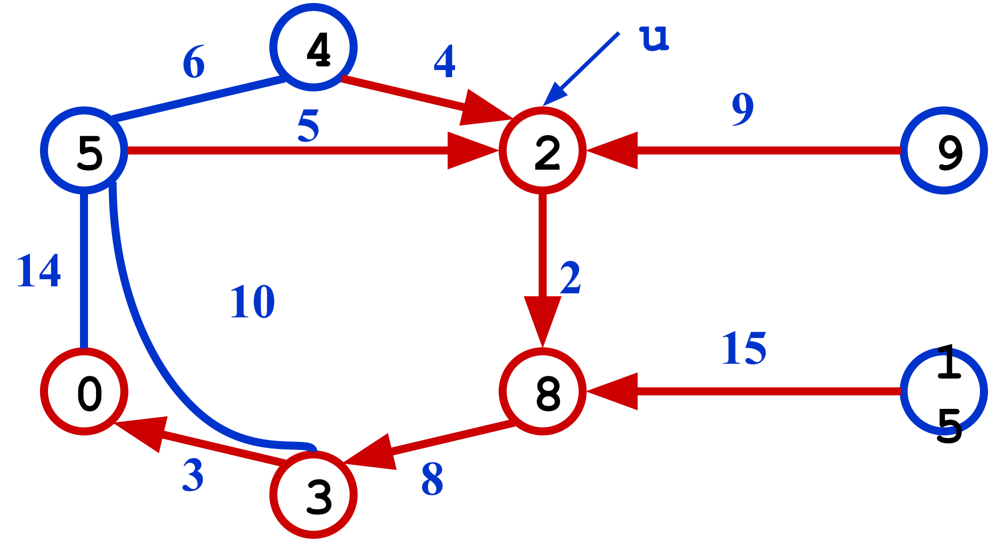

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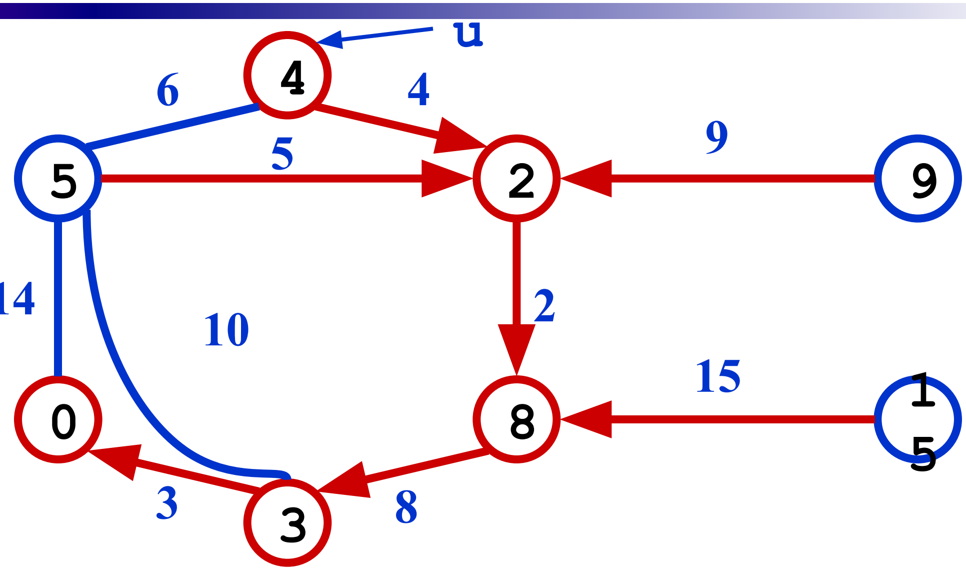

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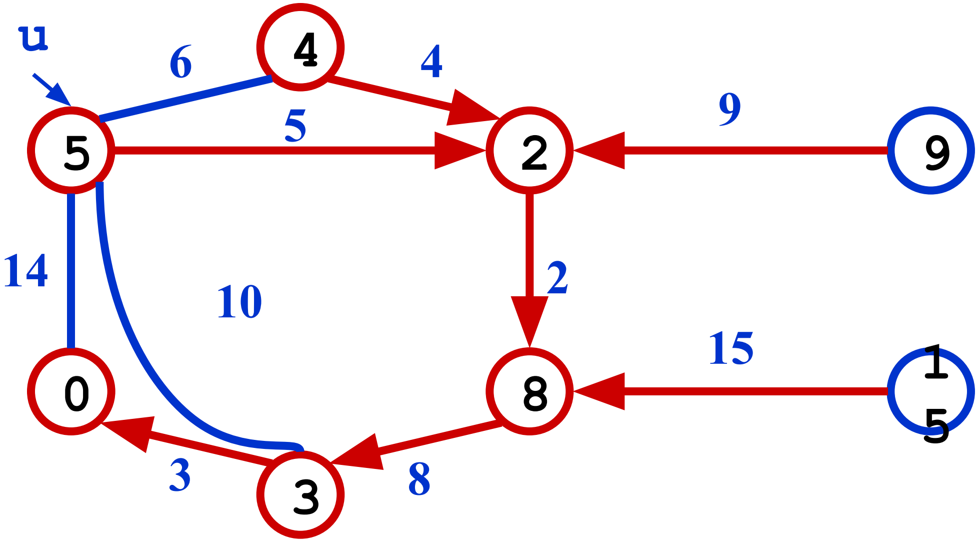

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    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```
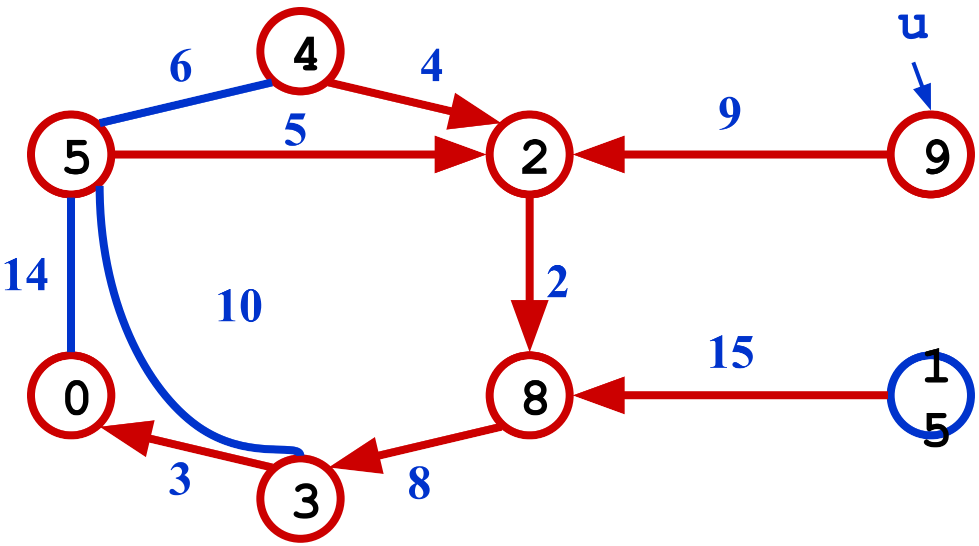
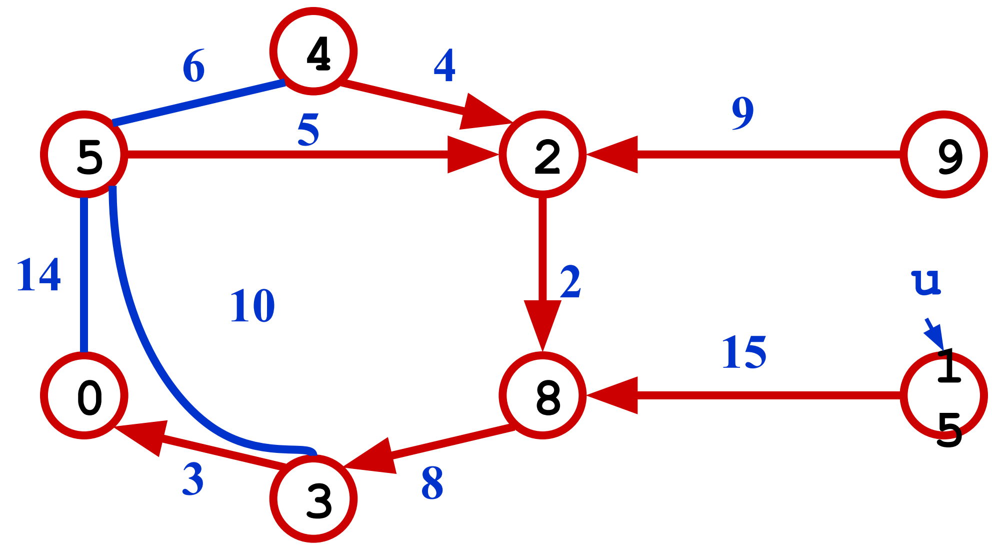



Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```



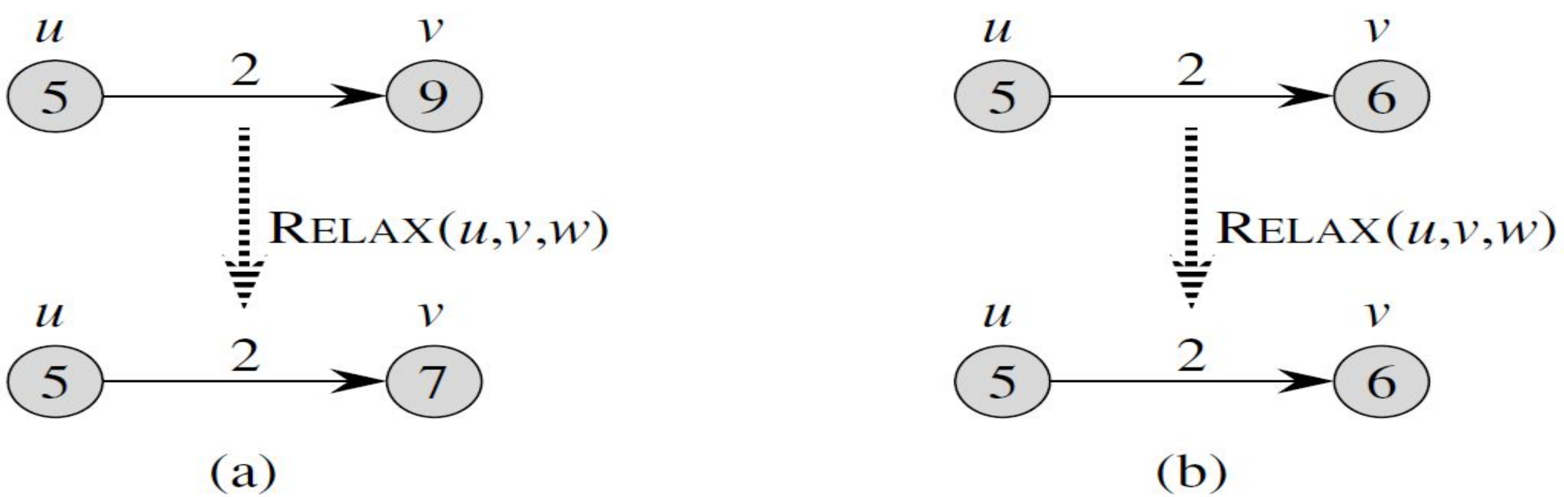
Single-Source Shortest Path

adapted from David Luebke

Relaxation

- A key technique in shortest path algorithms is *relaxation*
  - Idea: for all *v*, maintain upper bound d[*v*] on δ(s,v)

```
Relax(u,v,w) {
    if (d[v] > d[u]+w) then d[v]=d[u]+w;
}
```



Relaxation

INITIALIZE-SINGLE-SOURCE(G, s)

for each vertex $v \in V[G]$
    do $d[v] \leftarrow \infty$
        $\pi[v] \leftarrow \text{NIL}$
$d[s] \leftarrow 0$

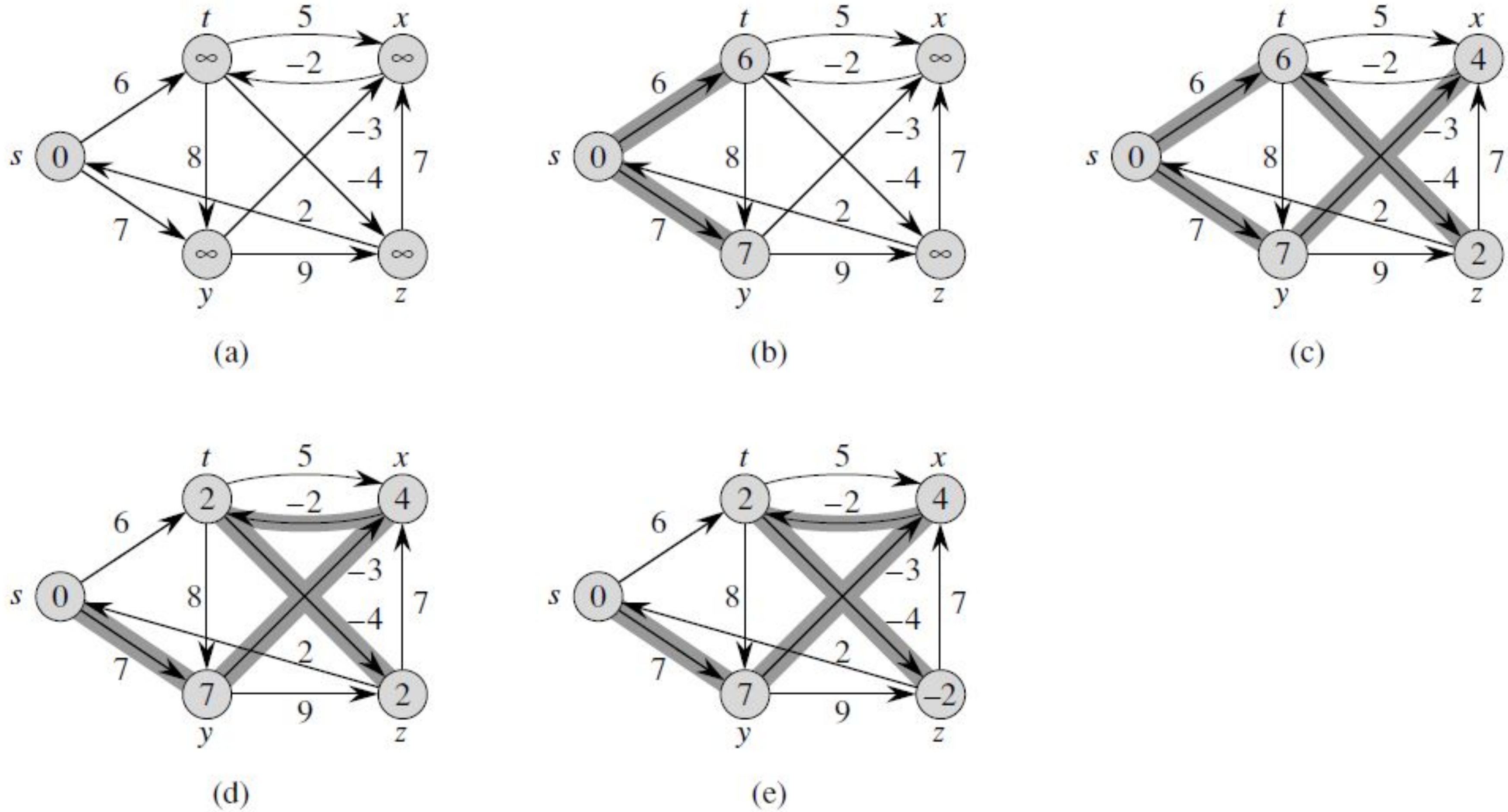
RELAX(u, v, w)

if $d[v] > d[u] + w(u, v)$
    then $d[v] \leftarrow d[u] + w(u, v)$
        $\pi[v] \leftarrow u$

Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
for i ← 1 to |V[G]| − 1
    do for each edge (u, v) ∈ E[G]
        do RELAX(u, v, w)
for each edge (u, v) ∈ E[G]
    do if d[v] > d[u] + w(u, v)
        then return FALSE
return TRUE
```

Bellman-Ford Example



Bellman-Ford

- Note that order in which edges are processed affects how quickly it converges
- Correctness: show $d[v] = \delta(s,v)$ after $|V|$-1 passes
  - Lemma: $d[v] \geq \delta(s,v)$ always
    - Initially true
    - Let v be first vertex for which $d[v] < \delta(s,v)$
    - Let u be the vertex that caused d[v] to change: $d[v] = d[u] + w(u,v)$
    - Then $d[v] < \delta(s,v)$
      $\delta(s,v) \leq \delta(s,u) + w(u,v)$ (*Why?*)
      $\delta(s,u) + w(u,v) \leq d[u] + w(u,v)$ (*Why?*)
    - So $d[v] < d[u] + w(u,v)$. Contradiction.

The End