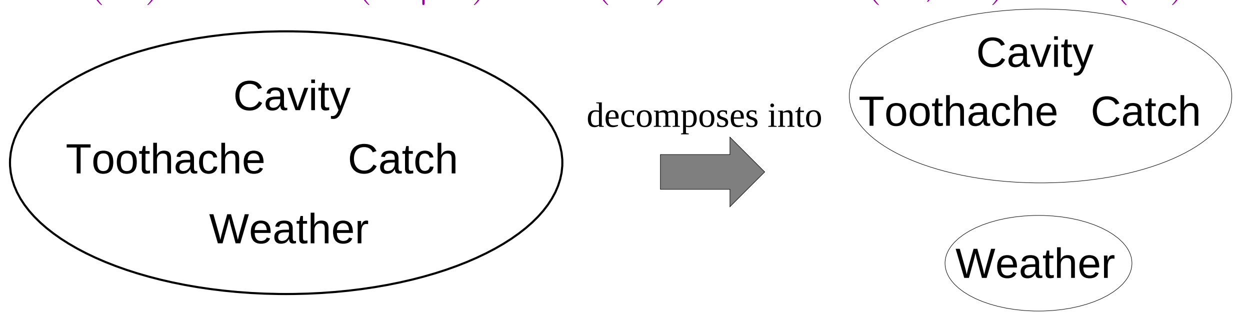
Inference by Enumeration

- Obvious problems:
  - Worst-case time complexity $O(d^n)$
  - Space complexity $O(d^n)$ to store the joint distribution

Independence

A and B are independent iff

$\mathbf{P}(A|B) = \mathbf{P}(A)$ or $\mathbf{P}(B|A) = \mathbf{P}(B)$ or $\mathbf{P}(A, B) = \mathbf{P}(A)\mathbf{P}(B)$



$$\mathbf{P}(Toothache, Catch, Cavity, Weather)$$
$$= \mathbf{P}(Toothache, Catch, Cavity)\mathbf{P}(Weather)$$

32 entries reduced to 12; for n independent biased coins, $2^n \to n$

Absolute independence powerful but rare

Dentistry is a large field with hundreds of variables,
none of which are independent. What to do?

Conditional independence

$\mathbf{P}(Toothache, Cavity, Catch)$ has $2^3 - 1 = 7$ independent entries

If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:

(1) $P(catch|toothache, cavity) = P(catch|cavity)$

The same independence holds if I haven't got a cavity:

(2) $P(catch|toothache, \neg cavity) = P(catch|\neg cavity)$

$Catch$ is conditionally independent of $Toothache$ given $Cavity$:

$\mathbf{P}(Catch|Toothache, Cavity) = \mathbf{P}(Catch|Cavity)$

Equivalent statements:

$\mathbf{P}(Toothache|Catch, Cavity) = \mathbf{P}(Toothache|Cavity)$

$\mathbf{P}(Toothache, Catch|Cavity) = \mathbf{P}(Toothache|Cavity)\mathbf{P}(Catch|Cavity)$

Conditional independence contd.

Write out full joint distribution using chain rule:

$$\mathbf{P}(Toothache, Catch, Cavity)$$
$$= \mathbf{P}(Toothache|Catch, Cavity)\mathbf{P}(Catch, Cavity)$$
$$= \mathbf{P}(Toothache|Catch, Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity)$$
$$= \mathbf{P}(Toothache|Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity)$$

I.e., $2 + 2 + 1 = 5$ independent numbers (equations 1 and 2 remove 2)

In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in n to linear in n.

Conditional independence is our most basic and robust form of knowledge about uncertain environments.

Bayes' Rule

Product rule $P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$

$\Rightarrow$ Bayes' rule $P(a|b) = \dfrac{P(b|a)P(a)}{P(b)}$

or in distribution form

$$\mathbf{P}(Y|X) = \frac{\mathbf{P}(X|Y)\mathbf{P}(Y)}{\mathbf{P}(X)} = \alpha\mathbf{P}(X|Y)\mathbf{P}(Y)$$

Useful for assessing diagnostic probability from causal probability:

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

E.g., let M be meningitis, S be stiff neck:

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.8 \times 0.0001}{0.1} = 0.0008$$

Note: posterior probability of meningitis still very small!

Quiz: Bayes' Rule

- Given:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

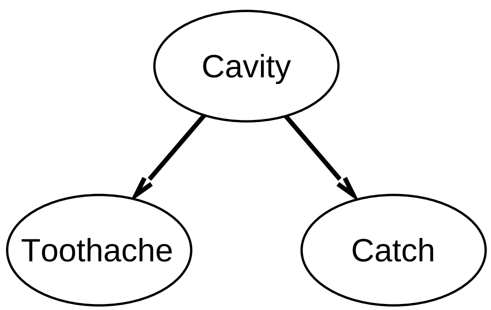
- What is P(W | dry) ?

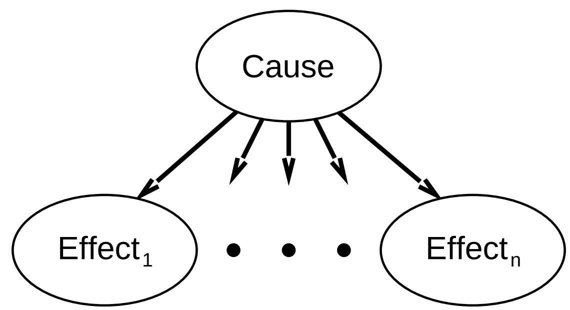
Bayes' Rule and conditional independence

$$\mathbf{P}(Cavity|toothache \wedge catch)$$
$$= \alpha\, \mathbf{P}(toothache \wedge catch|Cavity)\mathbf{P}(Cavity)$$
$$= \alpha\, \mathbf{P}(toothache|Cavity)\mathbf{P}(catch|Cavity)\mathbf{P}(Cavity)$$

This is an example of a naive Bayes model:

$$\mathbf{P}(Cause, Effect_1, \ldots, Effect_n) = \mathbf{P}(Cause)\Pi_i \mathbf{P}(Effect_i|Cause)$$





Total number of parameters is **linear** in n

Today – Part 2: Probabilistic Reasoning

- Bayesian networks
  - Systematic way to represent the independence and conditional independence relationships.

Outline

◇ Syntax

◇ Semantics

◇ Parameterized distributions

Bayesian networks

A simple, graphical notation for conditional independence assertions and hence for compact specification of full joint distributions

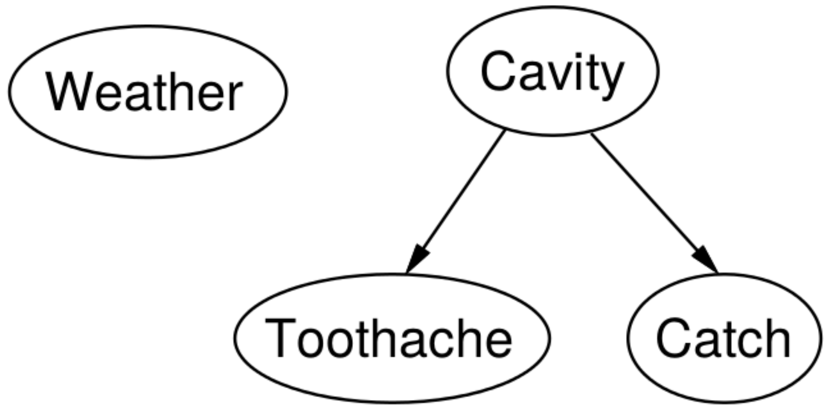
Syntax:

a set of nodes, one per variable
a directed, acyclic graph (link $\approx$ "directly influences")
a conditional distribution for each node given its parents:
$\mathbf{P}(X_i|Parents(X_i))$

In the simplest case, conditional distribution represented as a conditional probability table (CPT) giving the distribution over X_i for each combination of parent values

Example

Topology of network encodes conditional independence assertions:



$Weather$ is independent of the other variables

$Toothache$ and $Catch$ are conditionally independent given $Cavity$

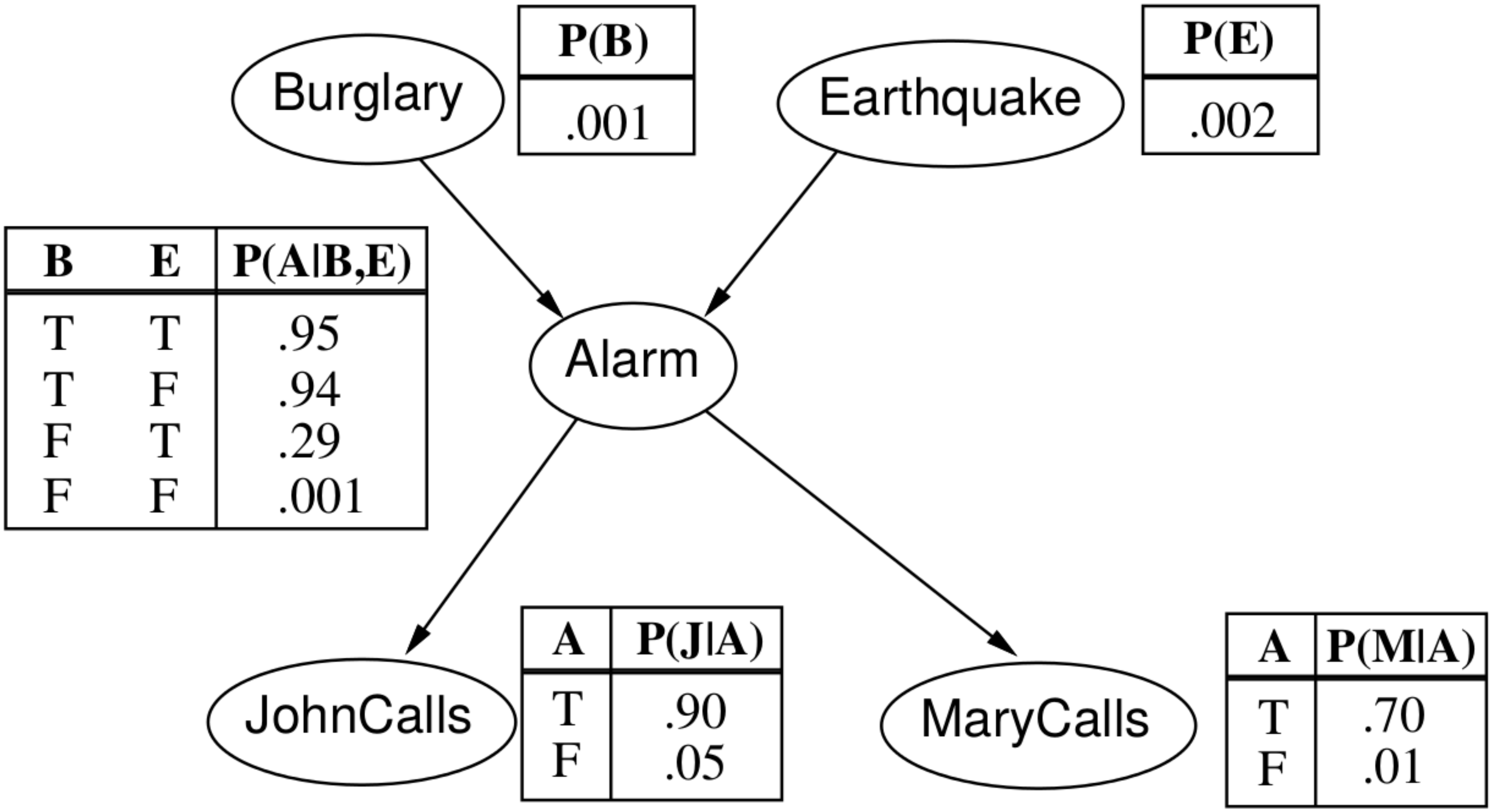
Example

I'm at work, neighbor John calls to say my alarm is ringing, but neighbor Mary doesn't call. Sometimes it's set off by minor earthquakes. Is there a burglar?

Variables: $Burglar$, $Earthquake$, $Alarm$, $JohnCalls$, $MaryCalls$
Network topology reflects "causal" knowledge:

- A burglar can set the alarm off
- An earthquake can set the alarm off
- The alarm can cause Mary to call -- Mary likes loud music so sometimes misses
- The alarm can cause John to call -- always calls, but sometimes confuses with telephone ringing,

Example contd.



P(B)
.001

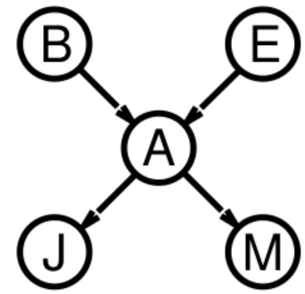
P(E)
.002

B	E	P(A|B,E)
T	T	.95
T	F	.94
F	T	.29
F	F	.001

A	P(J|A)
T	.90
F	.05

A	P(M|A)
T	.70
F	.01

Compactness



A CPT for Boolean X_i with k Boolean parents has
rows for the combinations of parent values

Each row requires one number p for $X_i = true$
(the number for $X_i = false$ is just $1 - p$)

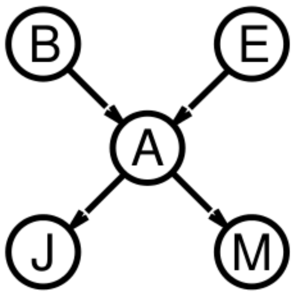
If each variable has no more than k parents,
the complete network requires $O(\quad)$ numbers

I.e., grows linearly with vs. $O(2^n)$ for the full joint distribution

For burglary net, $1 + 1 + 4 + 2 + 2 = 10$ numbers (vs. $2^5 - 1 = 31$)

Compactness

A CPT for Boolean X_i with k Boolean parents has 2^k rows for the combinations of parent values



Each row requires one number p for $X_i = true$ (the number for $X_i = false$ is just $1 - p$)

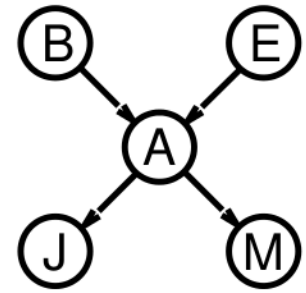
If each variable has no more than k parents, the complete network requires $O(n \cdot 2^k)$ numbers

I.e., grows linearly with n, vs. $O(2^n)$ for the full joint distribution

For burglary net, $1 + 1 + 4 + 2 + 2 = 10$ numbers (vs. $2^5 - 1 = 31$)

Global semantics

Global semantics defines the full joint distribution as the product of the local conditional distributions:



$$P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

e.g., $P(j \wedge m \wedge a \wedge \neg b \wedge \neg e)$

$=$ recall chain rule and conditional independence.

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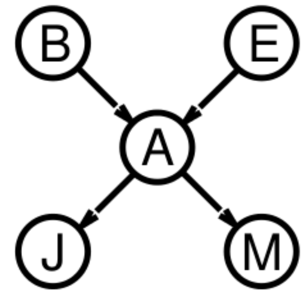
Global semantics

"Global" semantics defines the full joint distribution as the product of the local conditional distributions:



$$P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

e.g., $P(j \wedge m \wedge a \wedge \neg b \wedge \neg e)$

$$\begin{aligned} &= P(j|a)P(m|a)P(a|\neg b, \neg e)P(\neg b)P(\neg e) \\ &= 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998 \\ &\approx 0.00063 \end{aligned}$$

Constructing Bayesian networks

Need a method such that a series of locally testable assertions of conditional independence guarantees the required global semantics

1. Choose an ordering of variables $X_1, \ldots, X_n$
2. For $i = 1$ to n
   add X_i to the network
   select parents from $X_1, \ldots, X_{i-1}$ such that
   $$\mathbf{P}(X_i|Parents(X_i)) = \mathbf{P}(X_i|X_1,\,\ldots,\,X_{i-1})$$

This choice of parents guarantees the global semantics:

$$\begin{aligned}\mathbf{P}(X_1, \ldots, X_n) &= \prod_{i=1}^{n}\mathbf{P}(X_i|X_1,\,\ldots,\,X_{i-1}) \quad \text{(chain rule)}\\ &= \prod_{i=1}^{n}\mathbf{P}(X_i|Parents(X_i)) \quad \text{(by construction)}\end{aligned}$$

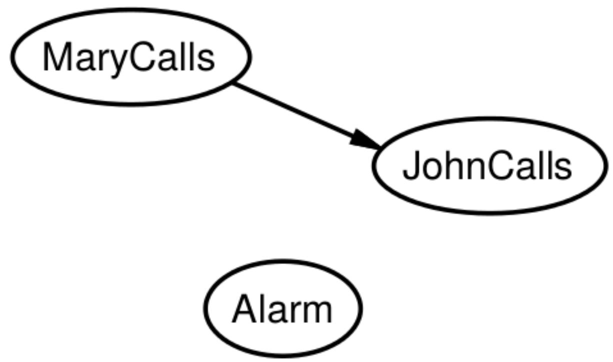
Example

Suppose we choose the ordering M, J, A, B, E

$P(J|M) = P(J)$?

Example

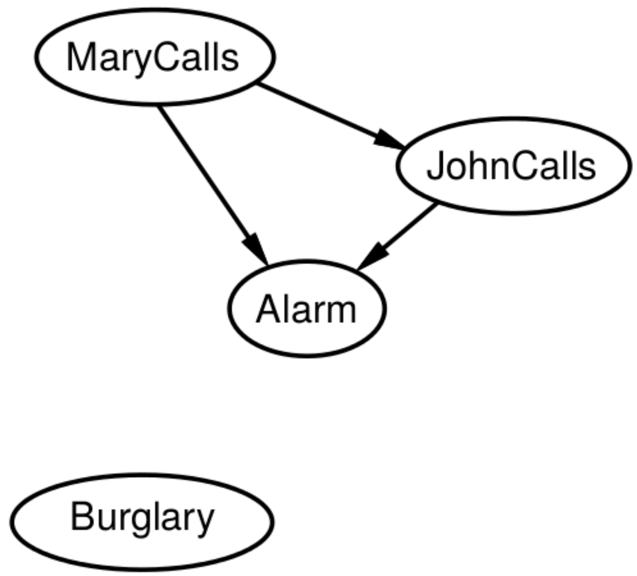
Suppose we choose the ordering M, J, A, B, E



$P(J|M) = P(J)$? No
$P(A|J,M) = P(A|J)$? $P(A|J,M) = P(A)$?

Example

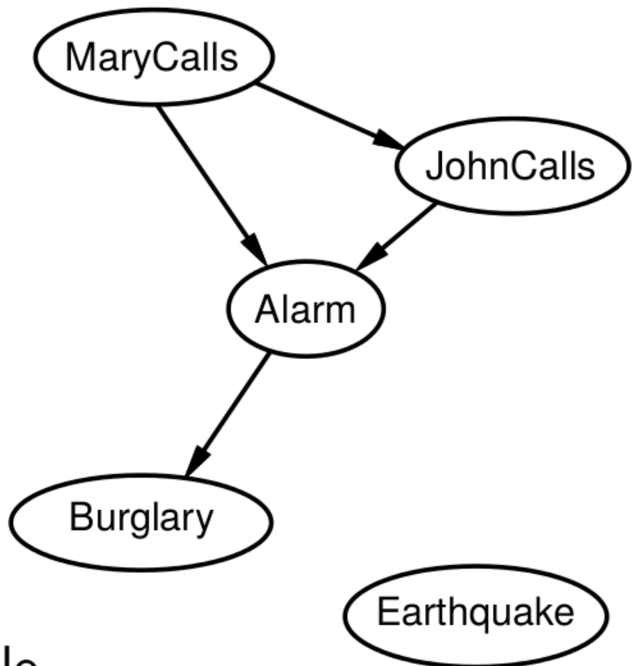
Suppose we choose the ordering M, J, A, B, E



$P(J|M) = P(J)$? No
$P(A|J,M) = P(A|J)$? $P(A|J,M) = P(A)$? No
$P(B|A,J,M) = P(B|A)$?
$P(B|A,J,M) = P(B)$?

Example

Suppose we choose the ordering M, J, A, B, E



$P(J|M) = P(J)$? No
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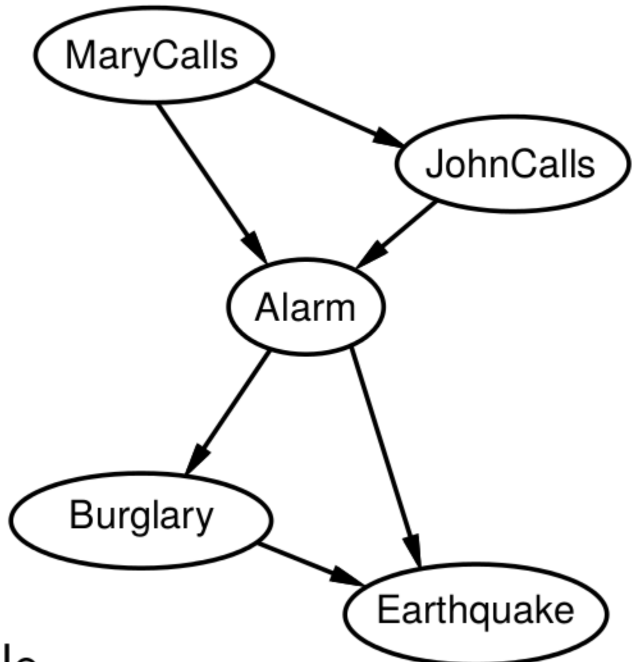
$P(B|A, J, M) = P(B|A)$? Yes
$P(B|A, J, M) = P(B)$? No
$P(E|B, A, J, M) = P(E|A)$?
$P(E|B, A, J, M) = P(E|A, B)$?

Example

Suppose we choose the ordering M, J, A, B, E



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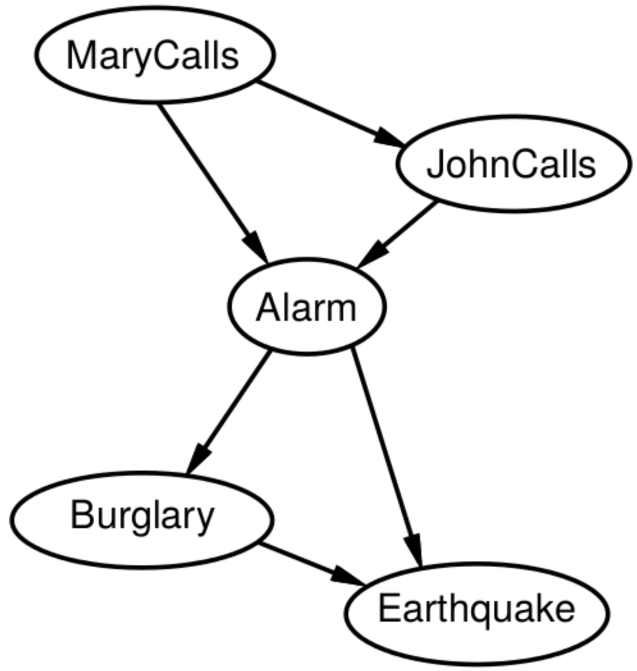
$P(B|A,J,M) = P(B|A)$? Yes
$P(B|A,J,M) = P(B)$? No
$P(E|B,A,J,M) = P(E|A)$? No
$P(E|B,A,J,M) = P(E|A,B)$? Yes

Example contd.



Deciding conditional independence is hard in noncausal directions

(Causal models and conditional independence seem hardwired for humans!)

Assessing conditional probabilities is hard in noncausal directions

Network is less compact: $1+2+4+2+4=13$ numbers needed